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# ON THE SCATTERED REFLECTION OF ELECTROMAGNETIC WAVES

by

VICTOR TWERSKY

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## ABSTRACT

The non-specular reflection of plane electromagnetic waves of arbitrary polarization by certain perfectly conducting surfaces composed of either semicylindrical or hemispherical bosses on an infinite plane is analyzed. Solutions in terms of eigenfunctions for the problem of the single boss on an infinite plane and a plane wave at an arbitrary angle of incidence are derived and extended, subject to the single scattering hypothesis, to obtain the far field solutions for certain small finite patterned distributions and both small finite and infinite uniform random distributions of bosses small compared to the wavelength. The results for the various cases are then compared in the plane of incidence and similarities between the analogous expressions for the distributions of semicylinders and hemispheres are noted. Expressions are obtained for the ratios of the reflected intensities and radial energy flux polarized parallel and perpendicular to the plane of incidence, as well as for the total intensity and radial energy flux for the case where the incident wave is unpolarized. It is found that for certain values of the parameters the reflected radiation may consist only of either the specular or the scattered contributions, while for other values of the parameters one of the scattered contributions, either parallel or perpendicular component may vanish. The results also indicate the occurrence of an extremum in the reflected radiation in the vicinity of the specular angle of reflection which for certain ranges of the parameters for the small finite distributions may be a minimum rather than a maximum. For these cases there is also some critical angle of incidence (not necessarily $\pi/2$ or grazing incidence) for which the reflection at the specular angle is completely specular.

The analogous distributions of cylinders and spheres are also considered.

## NOTATION

$\underline{E}$, $\underline{H}$, - The electric and magnetic field intensities which comprise the electromagnetic field.

i - Superscript or subscript characterizing the incident wave.

r - Superscript or subscript characterizing the total reflected wave.

p - Superscript or subscript characterizing the specularly reflected wave components.

c, s - Superscript or subscript characterizing the scattered components of the total reflected wave for the (semi)cylindrical and (hemi)spherical cases respectively.

A, B,- Arbitrary constants (determined from proscribed initial conditions) that specify the polarization parallel and perpendicular to the plane of incidence.

II', II" - Vector Hertz potentials of the electric and magnetic type respectively.

$\psi'$, $\psi''$ - Scalar components of the Hertz potentials.

$\underline{K} = K\hat{n} = (2\pi/\lambda)\hat{n} = (\omega/c)\hat{n}$ - Propagation vector (the symbol $\wedge$ denotes the unit vector).

$\alpha, \beta$ , - Angles of incidence.

$\hat{i}, \hat{j}, \hat{k}$ - Unit rectangular coordinate vectors.

$\underline{R} = r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) + z\hat{k}$ - Position vector in cylindrical coordinates.

A', B' - Constants introduced for brevity.

a - Radius of the boss.

$\varepsilon_n$ - Neumann's factor: $\varepsilon_n = 1$ if $n = 0$, $\varepsilon_n = 2$ if $n > 0$

$J_n$ - Bessel functions.

$H_n$ - Hankel functions of the first kind.

$Z_n'$ - Derivatives of the cylindrical functions with respect to their arguments.

$A_n'$, $A_n''$ - Cylindrical scattering coefficients.

o - Left superscript denoting the approximate solution ($Kr \gg 1$, $Ka < 1$) for the single boss at the origin of the reference coordinates.

Q (or K), q - Abbreviations for the amplitude and phase of the scattered components.

$\underline{F}$ - Either $\underline{E}$ or $\underline{H}$.

$q$ - Summation index ranging from $-N$ to $+N$.

$\mathcal{N}$ - Total number of bosses in a distribution.

$\Gamma$ - Fraunhofer phase factor.

$j$ - Subscript labeling the superposed distributions.

$\eta_j$ - Factor defined so that $\eta_j = 0$ for the particular value associated with the bosses of largest radii and $\eta_j = 1$ for all other values.

$M$ - Density function equal to the number of bosses per square centimeter.

$d$ - Linear extent of the small finite distributions.

$C$, $D$ - Constant coordinates of a field point for the infinite distributions.

$V$ - Volumetric departure from the plane equal to the total volume of bosses per square centimeter of distribution.

$\underline{r} = r(\sin\theta\cos\varphi\,\hat{i} + \sin\theta\sin\varphi\,\hat{j} + \cos\theta\,\hat{k})$ - Position vector in spherical coordinates.

$\left.\begin{array}{l} j_n \\ h_n \\ z_n' \end{array}\right\}$ - Spherical Bessel functions - Morse's notation.

$D_r$, etc. - Partial derivative with respect to $r$, etc.

$\Pi_{nm}$, $G'_{nm}$, $G''_{nm}$, $P'_{nm}$, $P''_{nm}$ - Symbols introduced for brevity.

$P_n^m(\cos\theta)_\theta$ - Derivatives of the Associated Legendre functions with respect to $\theta$.

$\sum_n \sum_m$, $\Sigma_n \Sigma_m$ - Sums over $n$ and $m$ such that $n$ and $m$ have odd or even parity respectively.

$a_n'$, $a_n''$ - Spherical scattering coefficients.

$\mu$ - Angle made by a line of bosses with the x-axis in the xy-plane: also used to label a particular line of bosses.

$\rho$, $\mu$ - Polar coordinates in xy-plane.

$\beta$ - Angle made by the axes of a rectangular distribution of spheres with the xy-axes: also used to label a particular rectangle.

u - "Reflection coefficient" for the infinite distributions.

$\gamma$ - Phase factor of u .

$\Delta$ - function - A function such as $\sin\Gamma/\Gamma$ or $\sin\eta\Gamma/\sin\Gamma$ which has a single principle maximum or a series of such maxima and which reduces to $\eta$ at the specular angle of reflection .

$J \equiv \mathrm{Re}(\underline{E}\cdot\underline{E}^*)$ - Quantity proportional to the intensity

$\rho \equiv J_{\parallel}/J_{\perp}$

S - Radial component of the complex Poynting vector.

$\sigma \equiv S_{\parallel}/S_{\perp}$

## 1. Introduction

The scattered reflection of plane scalar waves from certain surfaces consisting of distributions of semicylindrical or hemispherical bosses (protuberances) on an infinite plane was analyzed in a previous report[1]. The present report considers the analogous problems for the reflection of electromagnetic waves from metallic surfaces of infinite conductivity.

The procedure we will follow will be to obtain solutions in terms of the appropriate eigenfunctions for the reflection problem of a plane electromagnetic wave of arbitrary polarization incident at some arbitrary angle on either a semicylindrical or hemispherical boss on an infinite plane. The solution for the single boss will then be extended, subject to the "single scattering" hypothesis, to obtain approximate far field solutions for certain small (as compared to the distance of observation) finite patterned distributions and both small finite and infinite uniform random distributions of bosses of radii small compared to the wavelength. A discussion of the derived results will be reserved for the last section where the analogous expressions for the cylindrical and spherical cases will be compared in the plane of incidence.

Except for the basic solutions to the problem of the single boss, the mathematical scope and content of this report do not differ appreciably from the previous treatment of the scalar cases. For this reason, while the derivation of the solutions to the problems of the single boss will be given in detail, only the significant differences between the present and previous procedures for deriving the results for the distributions will be included--the manipulations required in going from the single boss to its distributions being very similar to those employed previously and leading eventually to essentially the same sums and integrals which were there evaluated. (Page references to the previous report will be found in the text to facilitate the discussion).

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[1] V. Twersky, On the Scattered Reflection of Scalar Waves from Absorbent Surfaces, New York University, Mathematics Research Group, Research Report No. EM-22, August, 1950.

The eigenfunction formulation employed for the problem of the single boss restricts us, at least for practical purposes, to the range where the radii of the bosses are small compared to the incident wavelength (or more strictly speaking where $Ka = 2\pi a/\lambda < 1$). Although the functions employed are tabulated and in principal the initially derived expressions are valid for all values of Ka, it is only for $Ka < 1$ that the series are sufficiently convergent to be approximated by their first few terms. The expressions for the distributions derived subject to this limitation are therefore only appropriate for the diffraction range of phenomena (or the "Rayleigh scattering" region) as might occur with surfaces whose irregularities are small compared to the wavelength. In this region the distributions of bosses on a plane present features characteristic of either striated or rough surfaces and should be useful in investigating some of the general aspects of non-specular reflection.[2]

The restriction of $Ka < 1$ of necessity precludes the discussion of a wide variety of phenomena. In particular it excludes the optical range where there is reason for believing that reflection is governed by phenomena arising because the usual surface irregularities are large compared to wavelength--essentially the region for which the empirical Lambert's Law of diffuse reflection is considered valid. A certain amount of theoretical work has been done in this range, particularly in an effort to derive Lambert's Law which is believed to be a statistical effect of specular reflections occurring at the irregularities themselves.[3] This phase of the problem will perhaps be investigated by the techniques of geometrical optics in a future report.

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[2] As was mentioned in the previous report, the solution for the simple semi-cylindrical boss had been obtained by Rayleigh, reference 6. It is interesting to note that another approach to such problems as reflection from a rough surface introduced by Rayleigh, "Theory of Sound", Section 272a, Dover, N.Y., (1945)--who considered the problem of a plane wave incident on a corrugated surface whose cross-section could be represented by a Fourier series--is being developed by C. T. Tai, "Reflection and Refraction of a Plane Electromagnetic Wave at a Periodical Surface", Harvard Tech. Report No. 28, Cruft Lab., (Jan. 1948) and S. O. Rice, "Wave Propagation Over Rough Surfaces", paper delivered at the Symposium on the Theory of Electromagnetic Waves, New York University, June, 1950.

The region $Ka < 1$, however, is one in which some interesting phenomena are predicted. It is here found, for example, that for certain ranges of the parameters the results are at variance with what might be surmised from an intuitive basis and yield a minimum for the reflected radiation observed in the vicinity of the specular angle of reflection.[4] They also predict the occurrence of certain critical angles of incidence for which the reflection at the specular angle is completely specular, and certain critical angles for which marked changes occur in the ratio of the reflected components polarized parallel and perpendicular to the plane of incidence. It would be of interest to subject this region ($Ka < 1$) to a critical experimental investigation to determine whether or not these predictions based on relatively simple models can be verified for actual surfaces.

## 2. Formulation of the Single Boss Problem and Some Preliminary Considerations

Consider a plane electromagnetic wave of arbitrary polarization incident at an arbitrary angle on a perfectly conducting surface consisting of either a semicylindrical or a hemispherical boss on an infinite plane. We seek solutions of Maxwell's equations that satisfy the boundary condition that the tangential component of the electric field intensity vanishes at the surface, and are subject to the usual Sommerfeld radiation condition that their scattered components reduce to outgoing waves at infinity. The incident polarization vector or electric field intensity, $\underline{E}^i$, is resolved into components parallel and perpendicular to the plane of incidence ($A\underline{E}^i$, $B\underline{E}^i$; the constants so introduced to be determined from prescribed initial conditions) and we seek to determine the reflected field specified by $\underline{E}^r$ and $\underline{H}^r$ such that $\underline{E} = \underline{E}^i + \underline{E}^r$ and $\underline{H} = \underline{H}^i + \underline{H}^r$ satisfy Maxwell's equations, the boundary condition $E_{\tan}\big|_{\text{surface}} = 0$, and the radiation condition.

---

[3] W. W. Barkas and R. F. S. Hearman, Phys. Soc. Proc., 51 274 (1939); W. A. Rense, J. Opt. Soc. Amer., 40 55 (1950); L. Ornestine and R. Van der Berg, Physica, 4 1181 (1937).

[4] Such minima at the specular angle of reflection were actually observed for the acoustical case with sound incident on surfaces whose irregularities were of the same order of magnitude as the wavelength. See E. Skudrzyk, Akustische Zeits., 4 189 (1939).

The problem of scattered reflection by the boss on the plane is soluble by an image method whereby the original problem is replaced by that of the scattering by either a cylinder or sphere of two incident waves--the original wave specified by $\underline{E}^i$ and its image in the removed plane specified by $\underline{E}^p$ (where $\underline{E}^p$ is determined from the boundary condition $(E^i_{\tan} + E^p_{\tan})\big|_{\text{plane}} = 0$)--and the determination of the scattered field $\underline{E}^{c,s}$. The configuration of Fig. 1a (or Fig. 2a) is replaced by that of Fig. 1b (or Fig. 2b) and we proceed to solve the image problem, since it can be shown that the solutions in regions bI are identical with those in regions aI, the required solutions, satisfying all the required conditions: i.e., that, for example, the electric field intensity in 1a, $\underline{E} = \underline{E}^i + \underline{E}^r$, is identical with that in 1b for $|\theta| \leqslant \pi/2$, $\underline{E} = \underline{E}^i + \underline{E}^p + \underline{E}^c$, so that the determination of $\underline{E}^c$ (the component scattered by the cylinder obtained by imposing the boundary condition $E_{\tan}\big|_{\text{cylinder}} = 0$, and so chosen so that it also satisfies the required boundary condition on the plane) will yield the scattered reflected component for the boss on the plane, $\underline{E}^r = \underline{E}^p + \underline{E}^c$.

We will follow the above procedure, but it will be found more convenient to formulate the problems in terms of suitable Hertz vector potentials from which the field may be readily derived. It can be shown[5] that the electromagnetic field in free space for a harmonic time factor $e^{-i\omega t}$, $\underline{E}$, $\underline{H}$, may be resolved into two partial fields $\underline{E}'$, $\underline{H}'$ and $\underline{E}''$, $\underline{H}''$, derivable from the Hertz vector potentials $\Pi'$, $\Pi''$, where the primed variables are of the "electric type" (associated with a distribution of electric dipoles) and the double primed variables of the "magnetic type" (associated with a distribution of magnetic dipoles). In mks units the relationship between the above quantities may be expressed as

$$(1) \qquad \underline{E}' = \nabla \times \nabla \times \Pi' = (K^2 + \nabla\nabla\cdot)\Pi' \,, \qquad \underline{H}' = \varepsilon \nabla \times \dot{\Pi}' = -iK\eta \nabla \times \Pi' \,,$$

$$(2) \qquad \underline{E}'' = -\mu \nabla \times \dot{\Pi}'' = i\mu\omega \nabla \times \Pi'' \,, \qquad \underline{H}'' = \nabla \times \nabla \times \Pi'' = (K^2 + \nabla\nabla\cdot)\Pi'' \,,$$

where $\Pi'$ and $\Pi''$ are solutions of the vector wave equation,

$$(3) \qquad \nabla \times \nabla \times \Pi - \nabla\nabla\cdot\Pi + \mu\varepsilon \ddot{\Pi} = (\nabla\times\nabla\times - \nabla\nabla\cdot - K^2)\Pi = 0 \,,$$

$$\left[ K^2 = (2\pi/\lambda)^2 = (\omega/c)^2 = \varepsilon\mu\omega^2, \ \eta = K/\mu\omega = (\varepsilon/\mu)^{\frac{1}{2}} \right]$$

where $\Pi$ in (3) stands for either $\Pi'$ or $\Pi''$. (For brevity and where no confusion will arise we will also in the future use $\Pi$ to denote either $\Pi'$ or $\Pi''$ and similarly with their associated constants).

For the problems with which we will be concerned, the scalar components of $\Pi$ will consist of sets of either the cylindrical or spherical eigenfunctions with arbitrary constants to be determined from the boundary and initial conditions. For the cylindrical case it will be convenient to choose the Hertz vectors of the form $\Pi = \psi \hat{k}$, parallel to the axis of the boss along z, while for the spherical case we will choose, $\Pi = \psi \underline{r}$, in the radial direction. For these choices of $\Pi$ it will then be recognized that the primed and double primed quantities in equations 1-3 are the perhaps more familiar TM (transverse magnetic) and TE (transverse electric) modes respectively; the term transverse meaning perpendicular to the vectorial direction of II and not necessarily to its direction of propagation, i.e., thus II' indicates that its associated magnetic field has no component parallel to $\hat{k}$ for the cylindrical case or no component parallel to $\underline{r}$ for the spherical case. In the following sections which are concerned with the mathematical analysis of the problems it will be seen that equations 1-3 simplify considerably because of this choice of the II.

---

[5] See for example, J. A. Stratton, Electromagnetic Theory, pp. 28-32, McGraw-Hill, N. Y., 1941.

## 3. The Semicylindrical Boss.

### A. The Single Boss.

Consider a plane electromagnetic wave characterized by $\underline{E}^i$ incident on a semicylindrical boss of radius-a with axis parallel to the z-axis on an infinite yz-plane as in Fig. 1a. We seek a solution of the problem (as formulated in 2 ) subject to the boundary condition that the tangential component of E vanishes on the plane and boss or that

$$(4) \qquad (E_z = E_{r=y} = 0)_{\theta=\pi/2} \quad , \quad (E_z = E_\theta = 0)_{r=a} \quad .$$

The image technique discussed in the preceding section will be employed to solve this problem.[6]

In general, the incident field can be initially decomposed into components parallel and perpendicular to the plane of incidence (as determined by the unit vectors $\hat{n}_1$ and $\hat{i}$), expressed in terms of the reference cartesian coordinates and then transformed to the cylindrical coordinates of the boss, as follows sequentially:

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[6] This technique was employed previously by Rayleigh, Phil. Mag., 14 350 (1907) to solve the above problem for the case of incidence perpendicular to the axis of the boss.

Figure 1a

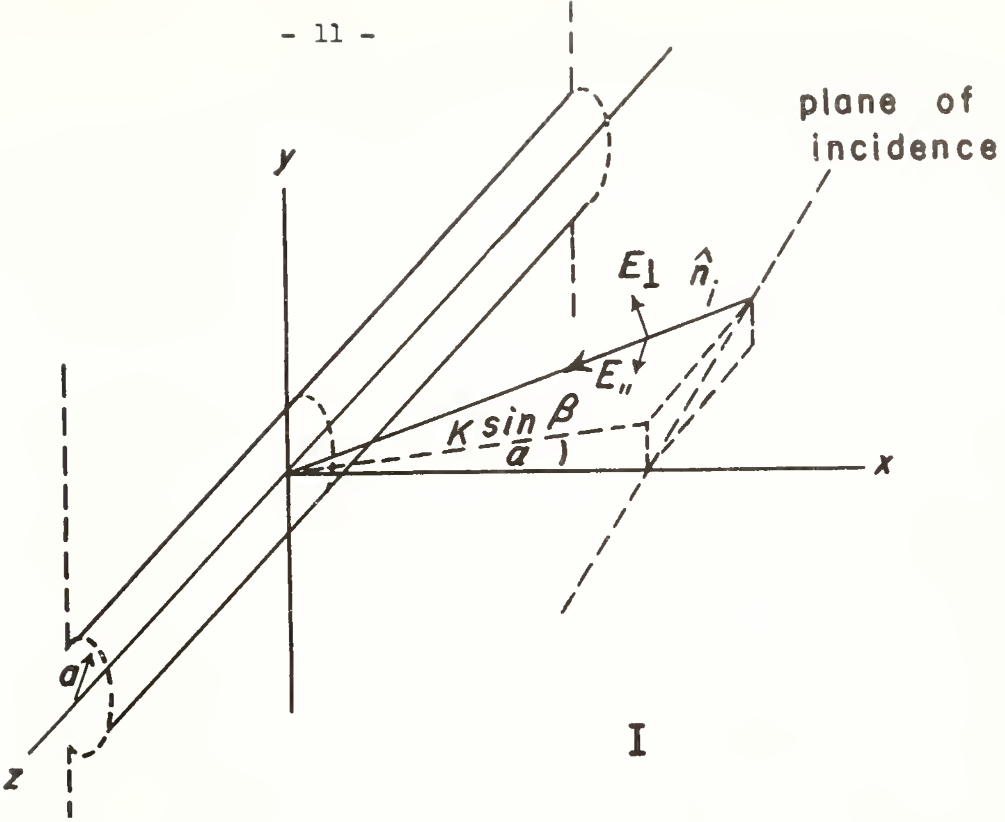
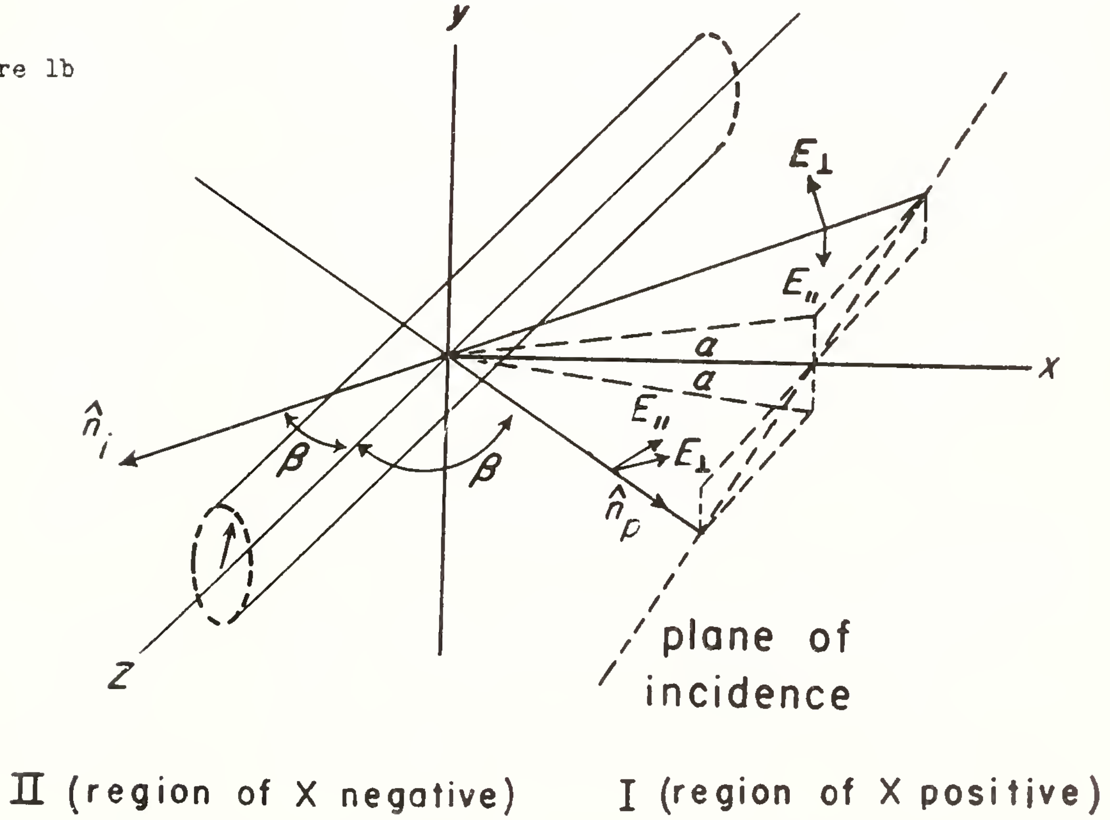


Figure 1b



a. A plane wave incident on a semicylindrical boss (whose axis is parallel to the z-axis) which lies on an infinite yz-plane. The plane of incidence is the plane of $\hat{n}_1$ and the x-axis. The unit vector $\hat{n}_1$ makes an angle $\beta$ with the z-axis while its trace on the xy-plane makes an angle $\alpha$ with the x-axis.

b. Two plane waves, $\hat{n}_1$ and its image in the yz-plane, $\hat{n}_p$, simultaneously incident on a circular cylinder of radius-a whose axis is parallel to the z-axis.

$$
\begin{aligned}
\underline{E}^{i} &= A\underline{E}^{i}_{\parallel} + B\underline{E}^{i}_{\perp} = \left\{ A\,\frac{(\hat{n}_i\times\hat{i})\times\hat{n}_i}{|(\hat{n}_i\times\hat{i})\times\hat{n}_i|} + B\,\frac{\hat{n}_i\times\hat{i}}{|\hat{n}_i\times\hat{i}|}\right\} e^{i\underline{K}_i\cdot\underline{R}} \\
&= \frac{e^{i\underline{K}_i\cdot\underline{R}}}{(1-\sin^2\beta\cos^2\alpha)^{\frac{1}{2}}}\Big\{ A\big[(1-\sin^2\beta\,\cos^2\alpha)\hat{i}-\sin^2\beta\,\cos\alpha\,\sin\alpha\,\hat{j}+\sin\beta\,\cos\beta\,\cos\alpha\,\hat{k}\big] \\
&\qquad + B\big[\cos\beta\,\hat{j}+\sin\beta\,\sin\alpha\,\hat{k}\big]\Big\} \\
&= \frac{e^{i\underline{K}_i\cdot\underline{R}}}{(1-\sin^2\beta\,\cos^2\alpha)^{\frac{1}{2}}}\Big\{ (A[\cos\theta - \sin^2\beta\,\cos\alpha\,\cos(\theta-\alpha)] + B\cos\beta\,\sin\theta)\,\hat{r} \\
&\qquad + (A[-\sin\theta + \sin^2\beta\,\cos\alpha\,\sin(\theta-\alpha) + B\cos\beta\,\cos\theta)\,\hat{\theta} \\
&\qquad + \sin\beta\,(A\cos\beta\,\cos\alpha + B\sin\alpha)\,\hat{k}\Big\} \quad ,
\end{aligned}
\tag{5}
$$

and

$$
\begin{aligned}
\underline{H}^{i} &= \frac{\underline{K}_i\times\underline{E}^{i}}{\mu\omega} = \eta\,\hat{n}_i\times\underline{E}^{i} = \eta\left\{-B(\hat{n}_i\times\hat{i})\times\hat{n}_i + A\hat{n}_i\times\hat{i}\right\}\frac{e^{i\underline{K}_i\cdot\underline{R}}}{(1-\sin^2\beta\,\cos^2\alpha)^{\frac{1}{2}}} \\
&= \frac{\eta\, e^{i\underline{K}_i\cdot\underline{R}}}{(1-\sin^2\beta\,\cos^2\alpha)^{\frac{1}{2}}}\Big\{ -B(1-\sin^2\beta\,\cos^2\alpha)\hat{i} + (B\sin^2\beta\,\cos\alpha\,\sin\alpha + A\cos\beta)\,\hat{j} \\
&\qquad + (-B\sin\beta\,\cos\beta\,\cos\alpha + A\sin\beta\,\sin\alpha)\,\hat{k}\Big\} \\
&= \frac{\eta\, e^{i\underline{K}_i\cdot\underline{R}}}{(1-\sin^2\beta\,\cos^2\alpha)^{\frac{1}{2}}}\Big\{ (-B[\cos\theta - \sin^2\beta\,\cos\alpha\,\cos(\theta-\alpha)] + A\cos\beta\,\sin\theta)\,\hat{r} \\
&\qquad + (-B[-\sin\theta + \sin^2\beta\,\cos\alpha\,\sin(\theta-\alpha)] + A\cos\beta\,\cos\theta)\,\hat{\theta} \\
&\qquad + \sin\beta\,(-B\cos\beta\,\cos\alpha + A\sin\alpha)\,\hat{k}\Big\} \quad .
\end{aligned}
$$

In these expressions the position vector of a field point P is $\underline{R} = r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) + z\hat{k} = \underline{r} + z\hat{k}$; the incident propagation vector is $\underline{K}_i = K\hat{n}_i = K(-\sin\beta\cos\alpha\,\hat{i} - \sin\beta\sin\alpha\,\hat{j} + \cos\beta\,\hat{k})$; and the transformations, $\hat{i} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$, $\hat{j} = \sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}$, $\hat{k} = \hat{k}$, have been employed. It can be seen that $\underline{H}^i$ may be obtained from $\underline{E}^i$ by replacing A by $-$B and B by A and multiplying the resulting expression by $\eta$.

The image wave in the yz-plane (characterized by $\underline{E}^p$ and $\underline{H}^p$) is the specularly reflected wave that would be obtained were the boss absent and is determined by imposing the first of conditions (4) on $\underline{E}^i + \underline{E}^p$. Thus on replacing $\underline{K}_i$ by the image propagation vector $\underline{K}_p = K\hat{n}_p = K(\sin\beta\cos\alpha\,\hat{i} - \sin\beta\sin\alpha\,\hat{j} + \cos\beta\,\hat{k})$ we obtain $\underline{E}^p$ by changing the signs of the $\hat{j}$ and $\hat{k}$ components of $\underline{E}^i$, and $\underline{H}^p$ by changing the sign of the $\hat{i}$ component of $\underline{H}^i$. Hence we obtain

$$\underline{E}^p = \frac{e^{i\underline{K}_p\cdot\underline{R}}}{(1-\sin^2\beta\cos^2\alpha)^{1/2}}\Big\{A(1-\sin^2\beta\cos^2\alpha)\hat{i} + (A\sin^2\beta\cos\alpha\sin\ -B\cos\beta)\hat{j}$$

$$-(A\sin\beta\cos\beta\cos\alpha + B\sin\beta\sin\alpha)\hat{k}\Big\}$$

(6)

$$= \frac{e^{i\underline{K}_p\cdot\underline{R}}}{(1-\sin^2\beta\cos^2\alpha)^{1/2}}\Big\{(A[\cos\theta - \sin^2\beta\cos\alpha\cos(\theta+\alpha)] - B\cos\beta\sin\theta)\hat{r}$$

$$+ (A[-\sin\theta + \sin^2\beta\cos\alpha\sin(\theta+\alpha)] - B\cos\beta\cos\theta)\hat{\theta}$$

$$- \sin\beta\,(A\cos\beta\cos\alpha + B\sin\alpha)\hat{k}\Big\}$$

and

$$\underline{H}^p = \eta\,\hat{n}_p\times\underline{E}^p = \frac{\eta\, e^{i\underline{K}_p\cdot\underline{R}}}{(1-\sin^2\beta\cos^2\alpha)^{1/2}}\Big\{B(1-\sin^2\beta\cos^2\alpha)\hat{i} + (B\sin^2\beta\cos\alpha\sin\alpha + A\cos\beta)$$

$$+ (-B\sin\beta\cos\beta\cos\alpha + A\sin\beta\sin\alpha)\hat{k}\Big\}$$

$$= \frac{\eta\, e^{i\underline{K}_p\cdot\underline{R}}}{(1-\sin^2\beta\cos^2\alpha)^{1/2}}\Big\{(B[\cos\theta - \sin^2\beta\cos\alpha\cos(\theta+\alpha)] + A\cos\beta\sin\theta)\hat{r}$$

$$+ (B[-\sin\theta + \sin^2\beta\cos\alpha\sin(\theta+\alpha)] + A\cos\beta\cos\theta)\hat{\theta}$$

$$+ \sin\beta\,(-B\cos\beta\cos\alpha + A\sin\alpha)\hat{k}\Big\}.$$

To proceed further and discover the fields $\underline{E}^c$ and $\underline{H}^c$ scattered by the cylinder subject to the boundary conditions at $r = a$, we now find it convenient to introduce two Hertz potentials of the form $\Pi = \psi\hat{k} = f(r, \theta)\, e^{ihz}\, \hat{k}$. Then, since $(\nabla\times\nabla\times - \nabla\nabla\cdot)\psi\hat{k} = -\nabla^2\psi\hat{k}$, equation (3) reduces to $(\nabla^2 + K^2)\psi = 0$, the scalar wave equation for a harmonic time dependence, so that the $\psi$ are the usual cylindrical eigenfunctions.[7] We therefore obtain from (1) and (2) that the field can be expressed as the sum of the two partial fields,

$$\begin{gathered}\underline{E}' = (K^2+\nabla\nabla\cdot)(\psi'\hat{k}) = K^2\psi'\hat{k} + ih\nabla\psi' = \left\{\hat{r}ih\frac{\partial}{\partial r} + \hat{\theta}\frac{ih}{r}\frac{\partial}{\partial\theta} + \hat{k}(K^2-h^2)\right\}\psi', \\ \underline{H}' = -iK\eta\nabla\times(\psi'\hat{k}) = iK\eta\hat{k}\times\nabla\psi' = iK\eta\left\{-\hat{r}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\theta}\frac{\partial}{\partial r}\right\}\psi',\end{gathered}\tag{7}$$

and

$$\begin{gathered}\underline{E}'' = i\mu\omega\nabla\times(\psi''\hat{k}) = i\mu\omega\left\{\hat{r}\frac{1}{r}\frac{\partial}{\partial\theta} - \hat{\theta}\frac{\partial}{\partial r}\right\}\psi'', \\ \underline{H}'' = (K^2+\nabla\nabla\cdot)(\psi''\hat{k}) = \left\{\hat{r}ih\frac{\partial}{\partial r} + \hat{\theta}\frac{ih}{r}\frac{\partial}{\partial\theta} + \hat{k}(K^2-h^2)\right\}\psi'';\end{gathered}\tag{8}$$

where $h = K\cos\beta$ and where

$$\begin{gathered}\underline{E}' = \underline{E}'^{i} + \underline{E}'^{p} + \underline{E}'^{c}, \qquad \underline{E}'' = \underline{E}''^{i} + \underline{E}''^{p} + \underline{E}''^{c}, \\ \underline{H}' = \underline{H}'^{i} + \underline{H}'^{p} + \underline{H}'^{c}, \qquad \underline{H}'' = \underline{H}''^{i} + \underline{H}''^{p} + \underline{H}''^{c}, \\ \psi' = \psi'_i + \psi'_p + \psi'_c, \quad \text{and} \quad \psi'' = \psi''_i + \psi''_p + \psi''_c;\end{gathered}\tag{9}$$

will now be determined.

It should be noted from (7) and (8) that $E_z = E'_z = (K^2 - h^2)\psi'$ and $H_z = H''_z = (K^2 - h^2)\psi''$. The following procedure hinges on these two expressions since the z components of the incident and image waves yield the corresponding $\psi$'s with which we determine the scattered $\psi$'s and hence the scattered field.

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[7] A discussion of the functions and expansions used in this section will be found in Stratton (reference 6), Ch. VI. Equations (7) and (8) will be found on pp. 360, 361.

The $\psi_i$ and $\psi_p$ are readily determined from the z components of (5) and (6). Thus (since $K^2 - h^2 = K^2\,(1-\cos^2\beta\,) = K^2\,\sin^2\beta\,$) we obtain

$$
\begin{aligned}
\psi'_i &= E^i_z/K^2\sin^2\beta = B'\,e^{i\underline{K}_i\cdot\underline{R}} ,\\
\psi'_p &= E^p_z/K^2\sin^2\beta = -B'\,e^{i\underline{K}_p\cdot\underline{R}} ,\\
\psi''_i &= H^i_z/K^2\sin^2\beta = A'\,e^{i\underline{K}_i\cdot\underline{R}} ,\\
\psi''_p &= H^p_z/K^2\sin^2\beta = A'\,e^{i\underline{K}_p\cdot\underline{R}} ;
\end{aligned}
\tag{10}
$$

where $B' = \dfrac{A\cos\beta\,\cos\alpha + B\sin\alpha}{K^2\sin\beta\,(1-\sin^2\beta\,\cos^2\alpha\,)^{\frac{1}{2}}}$ and $A' = \eta\,\dfrac{-B\cos\beta\,\cos\alpha + A\sin\alpha}{K^2\sin\beta\,(1-\sin^2\beta\,\cos^2\alpha\,)^{\frac{1}{2}}}$ .

[It should be noted that for incidence perpendicular to the axis of the boss, $\beta = \pi/2$ so that $B' = B/K^2$ and $A' = \eta\,A/K^2$. For this case, the primed quantities (the TM mode) correspond to a wave polarized parallel to the z-axis while the doubly primed quantities (the TE mode) correspond to a wave polarized perpendicular to the z-axis. Even for $\beta \neq \pi/2$, the problem could of course be formulated initially in terms of polarization components parallel and perpendicular to z (as is usually done for problems involving cylinders), but this involves some loss of generality (for the reflection problem of the boss on the plane) as compared to the adopted procedure of referring the polarization to the plane of incidence.]

We are now in a position to solve the problem of Fig. 1b subject to the second condition of (4); i.e., we consider the original wave and its image simultaneously incident on the cylinder and seek to determine the scattered field $\underline{E}^c$, $\underline{H}^c$, by first deriving the corresponding $\psi_c$. To facilitate imposing the boundary condition at $r = a$, we first expand the potentials in terms of cylindrical functions around the axis of the cylinder. Now

$$e^{i\underline{K}_i\cdot\underline{R}} = e^{-iKr\sin\beta\cos(\theta-\alpha)+iKz\cos\beta} = e^{-iK'r\cos(\theta-\alpha)+ihz}$$

$$= e^{ihz}\sum_{n=0}^{\infty} i^n \varepsilon_n J_n(K'r)\cos n(\pi-\theta+\alpha) \quad ,$$

where the $J_n$ are Bessel functions, and $\varepsilon_n = 1$ if $n = 0$ and $\varepsilon_n = 2$ if $n > 0$.

Similarly

$$e^{i\underline{K}_p\cdot\underline{R}} = e^{iKr\sin\beta\cos(\theta+\alpha)+iKz\cos\beta} = e^{ihz}\sum_{n=0}^{\infty} i^n\varepsilon_n J_n(K'r)\cos n(\theta+\alpha) .$$

Hence

$$\psi_i' + \psi_p' = B'e^{ihz}\sum i^n\varepsilon_n J_n(K'r)\left\{\cos n(\pi-\theta+\alpha) - \cos n(\theta+\alpha)\right\} \quad ,$$

(11)

$$\psi_i'' + \psi_p'' = A'e^{ihz}\sum i^n\varepsilon_n J_n(K'r)\left\{\cos n(\pi-\theta+\alpha) + \cos n(\theta+\alpha)\right\} \quad .$$

When these plane waves are incident on the cylinder[8] the corresponding scattered waves may be expressed in the same form as (11) but with $J_n$ replaced by $A_n H_n$; where the $H_n$, the Hankel functions of the first kind, are associated with outgoing cylindrical waves (because of the suppressed time factor $e^{-i\omega t}$). We therefore write

$$\psi' = \psi_i' + \psi_p' + \psi_c' = B'e^{ihz}\sum i^n\varepsilon_n\left[J_n + A_n'H_n\right]\left\{\cos n(\pi-\theta+\alpha) - \cos n(\theta+\alpha)\right\} \quad ,$$

(12)

$$\psi'' = \psi_i'' + \psi_p'' + \psi_c'' = A'e^{ihz}\sum i^n\varepsilon_n\left[J_n + A_n''H_n\right]\left\{\cos n(\pi-\theta+\alpha) + \cos n(\theta+\alpha)\right\} .$$

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[8] See pp. 6,7 in reference 1. The problem of a plane electromagnetic wave incident on a cylinder will be found in Frank-Mises, Differentialgleichungen der Physik, pp. 869-871, Rosenberg, N.Y., 1943.

The constants $A_n'$, $A_n''$, are now chosen to satisfy the boundary condition $(E_z = E_\theta = 0)_{r=a}$. From (7) and (8) it can be seen that $E_z' \propto \psi'$, $E_\theta' \propto \frac{\partial \psi'}{\partial \theta}$, while $E_z'' = 0$, $E_\theta'' \propto \frac{\partial \psi''}{\partial r}$, so that the restrictions on the $\psi$'s are that $(\psi' = \frac{\partial \psi''}{\partial r} = 0)_{r=a}$. Imposing these on (12) (i.e., replacing r by a in $\psi'$ and in $\frac{\partial \psi''}{\partial r}$ and then equating the coefficients of the trigonometric terms of each to zero) yields

$$A_n' = - J_n/H_n \quad , \quad A_n'' = - J_n'/H_n' \quad , \tag{13}$$

where $Ka \sin\beta = K'a$ is the argument of the Bessel functions.

Consequently (12) and the field derivable by means of (7) and (8) represent the solution to the image problem of Fig. 1b, and it can be seen that this is also the solution to the boss problem in the region $|\theta| \leq \pi/2$ satisfying the required boundary conditions at $r = a$ and $\theta = \pi/2$. The latter is of course obvious from (12) where it can be seen that $\psi_c$ has the same angular dependence as $\psi_i + \psi_p$ and must therefore satisfy the same boundary condition at $\theta = \pi/2$ since this condition depends only on θ or derivatives with respect to θ. To see analytically that the condition $(E_z = E_r = 0)_{\theta = /2}$ is satisfied, we note from (7) and (8) that $E_z' \propto \psi'$, $E_r' \propto \frac{\partial \psi'}{\partial r}$ while $E_z'' = 0$, $E_r'' \propto \frac{\partial \psi''}{\partial \theta}$. Hence $E_z'$ and $E_r' \propto \sin n(\pi/2 + \alpha) \sin n(\pi/2 - \theta)$ while $E_r'' \propto \frac{\partial}{\partial \theta} \cos n(\pi/2 + \alpha) \cos n(\pi/2 - \theta)$, so that all three quantities are zero at $\theta = \pi/2$ as is required.

Hence when a plane electromagnetic wave characterized by $\underline{E}^i$ as in (5) is incident on the boss on the infinite plane the scattered reflected Hertz potentials are

$$\psi_r' = \psi_p' + \psi_c' = \psi_p' - B'e^{ihz}\, 4 \sum_1^\infty i^n A_n' H_n \sin n(\pi/2 + \alpha) \sin n(\pi/2 - \theta) \; ,$$

(14)

$$\psi_r'' = \psi_p'' + \psi_c'' = \psi_p'' + A'e^{ihz}\, 2 \sum_0^\infty i^n \varepsilon_n A_n'' H_n \cos n(\pi/2 + \alpha) \cos n(\pi/2 - \theta) \; .$$

The total reflected electromagnetic field is then

$$\underline{E}^r = \underline{E}^p + \underline{E}^c , \tag{15}$$

where by means of equations (7) and (8)

$$\underline{E}^c=\underline{E}'^c+\underline{E}''^c=\left(iK \cos\beta \frac{\partial \psi_c'}{\partial r} + i \frac{\mu\omega}{r}\frac{\partial \psi_c''}{\partial\theta}\right)\hat{r}+\left(\frac{iK \cos\beta}{r}\frac{\partial \psi_c'}{\partial\theta} - i\mu\omega\frac{\partial \psi_c''}{\partial r}\right)\hat{\theta}+K^2\sin^2\beta\,\psi_c'\hat{k} . \tag{15a}$$

Making use of (14) we obtain

$$\begin{aligned}\underline{E}^c= \frac{e^{iKz \cos\beta}}{(1-\sin^2\beta\cos^2\alpha)^{\frac{1}{2}}}\Big\{&\Big[\cos\beta(A \cos\beta\cos\alpha+B \sin\alpha)4 \sum i^{n-1} A_n'H_n' \sin n(\pi/2+\alpha)\sin n(\pi/2- \theta)\\ &-([-B \cos\beta \cos\alpha+ A \sin\alpha] /Kr \sin\beta )4 \sum i^{n-1} A_n''H_n\, n \cos n(\pi/2+\alpha)\sin n(\pi/2- \theta)\Big]\hat{r}\\ &+\Big[ \cot\beta([A \cos\beta\cos\alpha+ B \sin\alpha] /Kr)4 \sum i^{n-1} A_n'H_n\, n \sin n (\pi/2 +\alpha ) \cos n(\pi/2- \theta)\\ &+(-B \cos\beta\cos\alpha+ A \sin\alpha )2 \sum i^{n-1}\varepsilon_n A_n''H_n' \cos n(\pi/2 +\alpha ) \cos n(\pi /2- \theta)\Big] \hat{\theta}\\ &-\Big[\sin\beta(A \cos\beta\cos\alpha+ B \sin\alpha )4 \sum i^n A_n'H_n \sin n(\pi /2+\alpha ) \sin n(\pi /2- \theta)\Big] \hat{k}\Big\} .\end{aligned} \tag{15b}$$

Similarly,

$$\underline{H}^r = \underline{H}^p + \underline{H}^c ,$$

where

$$\underline{H}^c =\underline{H}'^c+ \underline{H}''^c =\left(- \frac{iK\eta}{r}\frac{\partial \psi_c'}{\partial\theta}+ iK \cos\beta \frac{\partial \psi_c''}{\partial r}\right)\hat{r}+ \left(iK\eta\frac{\partial \psi_c'}{\partial r}+i\frac{K\cos\beta}{r} \frac{\partial \psi_c''}{\partial\theta}\right)\hat{\theta}+K^2\sin^2\beta\,\psi_c'' \hat{k} . \tag{15c}$$

Making use of (14) again we obtain

$$
\begin{aligned}
\underline{H}^c = {} & \eta \frac{e^{iKz\cos\beta}}{(1-\sin^2\beta\cos^2\alpha)^{\frac{1}{2}}} \\
& \times \Big\{ \Big[ ([A\cos\beta\cos\alpha + B\sin\alpha]/Kr\sin\beta) 4 \sum i^{n-1} A'_n H_n n \sin n(\pi/2+\alpha) \cos n(\pi/2-\theta) \\
& \quad -\cos\beta(-B\cos\beta\cos\alpha + A\sin\alpha) 2 \sum i^{n-1} \varepsilon_n A''_n H'_n \cos n(\pi/2+\alpha)\cos n(\pi/2-\theta) \Big] \hat{r} \\
& + \Big[ (A\cos\beta\cos\alpha + B\sin\alpha) 4 \sum i^{n-1} A'_n H'_n \sin n(\pi/2+\alpha) \sin n(\pi/2-\theta) \\
& \quad - \cot\beta([-B\cos\beta\cos\alpha + A\sin\alpha]/Kr) 4 \sum i^{n-1} A''_n H_n \, n \cos n(\pi/2+\alpha) \sin n(\pi/2-\theta) \Big] \hat{\theta} \\
& + \Big[ \sin\beta(-B\cos\beta\cos\alpha + A\sin\alpha) 2 \sum i^n \varepsilon_n A''_n H_n \cos n(\pi/2+\alpha)\cos n(\pi/2-\theta) \Big] \hat{k} \Big\} ,
\end{aligned}
\tag{15d}
$$

where $\underline{E}^p$ and $\underline{H}^p$ are given by equation (6) and where $\sum$ goes from 1 to $\infty$ while $\sum \varepsilon_n$ goes from 0 to $\infty$. The above constitutes the solution to the problem for the completely general case.

For $K'r \gg 1$ and $\gg n$, we can substitute the asymptotic form of the Hankel functions into the above series, i.e., $H_n \to (2/i\pi K'r)^{\frac{1}{2}} i^{-n} e^{iK'r}$, $H'_n \to (2/i\pi K'r)^{\frac{1}{2}} i^{-n+1} e^{iK'r}$. If $K'a<1$, the series are rapidly convergent and may be approximated by their first few terms because of the nature of the dependence of the coefficients $A'_n$, $A''_n$ on $K'a$, i.e., $A'_0 \approx i\pi/2 \ln(\gamma K'a/2)$, $A'_1 \approx -i\pi(K'a)^2/4$, $A'_n \approx -i\pi n(K'a/2)^{2n}/(n!)^2$; $A''_0 \approx -i\pi(K'a)^2/4$, $A''_1 \approx i\pi(K'a)^2/4$, $A''_n \approx i\pi n(K'a/2)^{2n}/(n!)^2$, (See reference 1, Appendix II). In $A'_0$, $\gamma = 1.781$ .

Hence to the order of $(K'a)^2/(K'r)^{1/2}$ the scattered quantities may be approximated by

$$\begin{aligned} {}^{0}\psi'_{c} &= B' e^{iKz\cos\beta} (\pi i/2K'r)^{\frac{1}{2}} (K'a)^{2} e^{iK'r}\, 2\cos\alpha\cos\theta \quad , \\ {}^{0}\psi''_{c} &= -A' e^{iKz\cos\beta} (\pi i/2K'r)^{\frac{1}{2}} (K'a)^{2} e^{iK'r} (1+2\sin\alpha\sin\theta) \quad , \end{aligned} \tag{16}$$

and

$$\begin{aligned} {}^{0}\underline{E}^{c} = -\frac{e^{i(Kz\cos\beta+K'r)}}{(1-\sin^{2}\beta\cos^{2}\alpha)^{\frac{1}{2}}} (\pi i/2K'r)^{\frac{1}{2}}(K'a)^{2} \Big\{ &\cos\beta(A\cos\beta\cos\alpha+B\sin\alpha)2\cos\alpha\cos\theta\,\hat{r} \\ &+(-B\cos\beta\cos\alpha+A\sin\alpha)(1+2\sin\alpha\sin\theta)\,\hat{\theta} \\ &-\sin\beta(A\cos\beta\cos\alpha+B\sin\alpha)2\cos\alpha\cos\theta\,\hat{k}\Big\} , \end{aligned} \tag{17}$$

$$ {}^{0}\underline{H}^{c} = -\eta\left\{\cos\beta\, {}^{0}E^{c}_{\theta}\,\hat{r} + ({}^{0}E^{c}_{z}/\sin\beta)\,\hat{\theta} - \sin\beta\, {}^{0}F^{c}_{\theta}\,\hat{k}\right\} \quad ,$$

where the left superscript 0 has been introduced to indicate that this is the approximate form for a boss located at the origin (or rather along the z-axis) of the reference coordinates (to facilitate the later extension of these results to distributions of bosses where the 0 will be replaced by a discrete index (c) or a continuous variable (y) and the expressions multiplied by a suitable phase factor chosen so that the phase at the reference origin of the reference coordinates is zero).

It can be seen that the formulation of the problem and the derived expressions (5,6,10,14,15,16 and 17) are considerably simplified when the wave is incident perpendicular to the axis of the boss, i.e., when $\beta = \pi/2$ and the xy-plane is the plane of incidence. Then the component $A\underline{F}^{i}_{\shortparallel}$ (parallel to the plane of incidence) is also parallel to the axis of the boss and the expressions are "separable in the polarizations"--i.e., the previous primed or TM quantities depend only on the

B polarization component of the incident wave which is now parallel to $\hat{k}$, the vectorial direction of the Hertz potential, while the double primed or TE quantities depend only on the A polarization component which is now perpendicular to $\hat{k}$.

For this case we obtain (on substituting $\beta = \pi/2$ in the previous expressions), the plane waves

$$(5') \qquad \underline{E}^i = \left\{-A[\sin(\theta-\alpha)\hat{r} + \cos(\theta-\alpha)\hat{\theta}] + B\hat{k}\right\} e^{-iKr\cos(\theta-\alpha)} ,$$

$$\underline{H}^i = \eta\left\{B[\sin(\theta-\alpha)\hat{r} + \cos(\theta-\alpha)\hat{\theta}] + A\hat{k}\right\} e^{-iKr\cos(\theta-\alpha)} ,$$

$$(6') \qquad \underline{E}^p = \left\{A[\sin(\theta+\alpha)\hat{r} + \cos(\theta+\alpha)\hat{\theta}] - B\hat{k}\right\} e^{iKr\cos(\theta+\alpha)} ,$$

$$\underline{H}^p = \eta\left\{B[\sin(\theta+\alpha)\hat{r} + \cos(\theta+\alpha)\hat{\theta}] + Ak\right\} e^{iKr\cos(\theta+\alpha)} ,$$

$$(10') \qquad \psi_i' = +(B/K^2)\, e^{-iKr\cos(\theta-\alpha)} , \quad \psi_p' = -(B/K^2)\, e^{iKr\cos(\theta+\alpha)} ,$$

$$\psi_i'' = \eta(A/K^2)\, e^{-iKr\cos(\theta-\alpha)} , \quad \psi_p'' = \eta(A/K^2)\, e^{iKr\cos(\theta+\alpha)} ;$$

the reflected potentials

$$(14') \quad \psi_r' = \psi_p' + \psi_c' = -(B/K^2)\left\{ e^{iKr\cos(\theta+\alpha)} + 4\sum i^n A_n' H_n \sin n(\pi/2+\alpha)\sin n(\pi/2-\theta)\right\} ,$$

$$\psi_r'' = \psi_p'' + \psi_c'' = \eta(A/K^2)\left\{ e^{iKr\cos(\theta+\alpha)} + 2\sum i^n \varepsilon_n A_n'' H_n \cos n(\pi/2+\alpha)\cos n(\pi/2-\theta)\right\}$$

(which, except for the multiplying factors, are identical with the velocity potentials derived for the analogous acoustical problems for the cases of 0 and ∞ surface impedance—see reference 1, page 8); and the reflected field

(15') $$\underline{E}^{r} = \underline{E}^{p} + \underline{E}^{c} \quad ,$$

$$\underline{E}^{c} = (E^{''c}_{r}\,\hat{r} + E^{''c}_{\theta}\,\theta) + E^{'c}_{k}\,k$$

$$= -(A/Kr)\; 4 \sum i^{n-1} A^{''}_{n} H_{n}\; n \cos n(\pi/2+\alpha) \sin n\,(\pi/2-\theta)\,\hat{r}$$

$$+ A\; 2 \sum i^{n-1} \varepsilon_{n} A^{''}_{n} H'_{n} \cos n(\pi/2+\alpha) \cos n(\pi/2-\theta)\,\hat{\theta}$$

$$- B\; 4 \sum i^{n} A'_{n} H_{n} \sin n(\pi/2+\alpha) \sin n(\pi/2-\theta)\;\hat{k} \quad ;$$

while

$$\underline{H}^{r} = \underline{H}^{p} + \underline{H}^{c} ,$$

$$\underline{H}^{c} = (H^{'c}\,\hat{r} + H^{'c}_{\theta}\,\hat{\theta}) + H^{''c}_{k}\,k$$

$$= \eta\,(B/Kr)\; 4 \sum i^{n-1} A'_{n} H_{n}\; n \sin n(\pi/2+\alpha) \cos n\,(\pi/2-\theta)\,\hat{r}$$

$$+ \eta\, B\; 4 \sum i^{n-1} A'_{n} H'_{n} \sin n(\pi/2+\alpha) \sin n\,(\pi/2-\theta)\,\hat{\theta}$$

$$+ \eta\, A\; 2 \sum i^{n} \varepsilon_{n} A^{''}_{n} H_{n} \cos n(\pi/2+\alpha) \cos n\,(\pi/2-\theta)\;\hat{k} \quad .$$

For Kr ≫ 1, Ka < 1, the scattered quantities reduce to

(16') $${}^{o}\psi'_{c} = (B/K^{2})\,(\pi\, i/2Kr)^{\frac{1}{2}}\,(Ka)^{2}\, e^{iKr}\, 2 \cos\alpha \cos\theta \quad ,$$

$${}^{o}\psi^{''}_{c} = -\eta\,(A/K^{2})(\pi\, i/2Kr)^{\frac{1}{2}}\,(Ka)^{2}\, e^{iKr}\,(1 + 2 \sin\alpha \sin\theta) \;,$$

(17') $${}^{o}\underline{E}^{c} = -\, e^{iKr}(\pi\, i/2Kr)^{\frac{1}{2}}(Ka)^{2}\Big[A(1+2 \sin\alpha \sin\theta)\hat{\theta} - B\; 2 \cos\alpha \cos\theta\;\hat{k}\Big] ,$$

$${}^{o}\underline{H}^{c} = \eta\big[-{}^{o}E^{c}_{k}\,\hat{\theta} + {}^{o}E^{c}_{\theta}\;\hat{k}\big] \quad .$$

(Expressions (17') are essentially those obtained by Rayleigh[6] for the above problem.)

On adding the p and c terms we obtain

$$(17'')\quad \underline{E}^r = e^{iKr\cos(\theta+\alpha)}\left\{ A\sin(\theta+\alpha)\,\hat{r} + A\left[\cos(\theta+\alpha) - Q_\theta\, e^{iq}\right]\hat{\theta} - B\left[1 - Q_k\, e^{iq}\right]\hat{k}\right\}$$

$$\underline{H}^r = \eta\, e^{iKr\cos(\theta+\alpha)}\left\{ B\sin(\theta+\alpha)\,\hat{r} + \left[B\cos(\theta+\alpha) - Q_k\, e^{iq}\right]\hat{\theta} + A\left[1 - Q_\theta\, e^{iq}\right]\hat{k}\right\}$$

where $q = Kr\left[1-\cos(\theta+\alpha)\right] + \pi/4 = 2Kr\sin^2\left[(\theta+\alpha)/2\right] + \pi/4,$

$$Q_\theta = (\pi/2Kr)^{\frac{1}{2}}\,(Ka)^2\left\{ (1+2\sin\alpha\sin\theta) + \mathcal{O}(Ka)^2 \ldots \right\},$$

$$Q_k = (\pi/2Kr)^{\frac{1}{2}}\,(Ka)^2\left\{ 2\cos\alpha\cos\theta + \mathcal{O}'(Ka)^2 \ldots \right\}.$$

(The notation $\mathcal{O}(Ka)^2$, $\mathcal{O}'(Ka)^2$, merely indicate that the next order terms in the curly brackets are to the order of $(Ka)^2$ or that the next order terms in Q are to the order of $(Ka)^4$).

Further consideration of the implications of the above will be reserved for the last section of this report.

B. Distributions of Bosses.

The asymptotic solutions obtained for the single boss can be extended to yield approximate solutions to some problems of scattered reflection from certain planar distributions of such bosses. The initial assumption is that the spacing of any two bosses is greater than the incident wavelength so that the excitation of each boss is essentially only the incident plane wave: i.e., the contributions to the excitation of each boss arising from the waves scattered by the remaining bosses is neglected--the so called "single scattering" hypothesis. (This will be further discussed in section 5.)

1) Patterned Distribution.

Consider a wave characterized by $\underline{E}^i$ (as in (5)) incident on $2N+1 = \mathcal{N}$ bosses of radius-a, whose axes lie in the yz-plane parallel to the z-axis and are spaced symmetrically along the y-axis with spacing-b--a reflection grating of bosses. Since the procedure for solving this problem is essentially the same as that of section IIB reference 1, it will be but briefly sketched here.[9]

We consider the point of intersection of the axis of each boss with the y-axis of our reference frame to be the origin (located at 0, qb, 0 in the xyz-reference frame of the central boss, q = 0) of a cylindrical coordinate system $r_q$, $\theta_q$, z, in which the distance of a field point P may be expressed as $\underline{R}_q = r_q (\cos\theta_q \hat{i} + \sin\theta_q \hat{j}) + z\hat{k} = r\cos\theta\,\hat{i} + r_q \sin\theta_q \hat{j} + z\hat{k}$. The Hertz potentials of the incident and image waves for the q'th boss differ from those for the central boss (as in (10)) only in that R is replaced by $R_q$ and the expressions multiplied by a phase factor $e^{-iK'qb\sin\alpha}$ (as obtained by setting $\theta = \pi/2$, $r = y = qb$, $z = 0$, in either $e^{i\underline{K}_i\cdot\underline{R}}$ or $e^{i\underline{K}_p\cdot\underline{R}}$) to correspond to the phase being zero at the central boss. Similarly the scattered potentials (the $\psi_c$ of (14) or (16)) differ only in possessing $r_q$ and $\theta_q$ (as will be indicated by writing ${}^q\psi_c$) and in being multiplied by the phase factor. Hence the scattered contributions of the entire distribution may be expressed as

$$\psi_c = \sum_{q=-N}^{N} {}^q\psi_c \, e^{-iK'qb\sin\alpha} \quad . \tag{18}$$

Now if the extent of the distribution is $\ll r$, (i.e., $\mathcal{N} b \ll r$) then we have approximately $r_q \approx r - qb\sin\alpha$, $\theta_q \approx \theta - (qb/r_q)\cos\theta \to \theta$. On embodying these approximations into the ${}^q\psi_c$ and substituting the asymptotic representation for the Hankel functions (neglecting $qb\sin\theta \ll r$ in the denominator) we find that the resulting sum over q is a geometrical progression which can be summed explicitly. If the restriction $K'a < 1$ is then invoked we obtain

---

[9] The procedure was originally applied to the problem of a plane wave normally incident on a grating of parallel cylinders by C. Schaefer and F. Reiche, Ann. d. Phys., 35 817 (1911).

(19) $$\psi_c = {}^{o}\psi_c \sin\mathcal{N}\Gamma/\sin\Gamma \quad , \Gamma = K'b(\sin\theta + \sin\alpha)/2 \quad ,$$

where the ${}^{o}\psi_c$ are as in (16) or (16'). We therefore obtain for the scattered reflected field components

(20) $$\underline{F}^r = \underline{F}^p + {}^{o}\underline{F}^c \sin\mathcal{N}\Gamma/\sin\Gamma \quad ,$$

where F = E or H and the appropriate quantities may be found in (6) and (17) or (6') and (17'). (That (19) implies (20) can be immediately verified from (15a,c) by noting that for $K'r \gg 1$ the $F^c$ depend essentially only on the $\psi$'s and their derivatives with respect to r; the derivatives with respect to $\theta$ which vanish as $r^{-3/2}$ being neglected. The quantities (20) could of course also be obtained by operating directly on the ${}^{o}F^c$ as in (18)-(19).)

The above may be readily extended to a superposition of several such reflection gratings--which are parallel and coplaner and overlap only at the central boss and differ in the number of elements ($\mathcal{N}$), their spacing (b), and their radii (a)--by summing the contributions of the various gratings. We would then obtain (20) but containing

(21) $$\sum_j a_j^2 \left[ (\sin\mathcal{N}_j\Gamma_j/\sin\Gamma_j) - \eta_j \right]$$

where $\eta_j = 0$ for one particular grating (that of largest $a_j$) and $\eta_j = 1$ for all the others to compensate for the overlap at the center. At the specular angle of reflection (essentially when $\theta = -\alpha$) the $\eta_j = 0$ for all j so that $\sin\mathcal{N}_j\Gamma_j/\sin\Gamma_j \rightarrow \mathcal{N}_j$ and (21) yields $\sum_j a_j^2(\mathcal{N}_j - \eta_j)$; which for all $a_j = a$ yields $a^2\sum(\mathcal{N}_j - \eta_j) = a^2\mathcal{N}$, where $\mathcal{N}$ is the total number of bosses in the superposed distribution. We therefore obtain

(22) $$(\underline{F}^r = \underline{F}^p + {}^{o}\underline{F}^c\mathcal{N})_{\theta = -\alpha} \quad .$$

This procedure may also be extended to consider overlaps occurring elsewhere than at the center, i.e., when the spacings of various gratings are commensurate, by compensating for the overlaps, as above, by the addition and the substraction of the missing terms. It can be seen that at the specular angle these terms are equal to 1 so that we again obtain (22). Also since this is true for any distribution which may be decomposed into patterned distributions, we may therefore conclude that this is also true for any distribution of parallel bosses of equal radii—subject to the initial restrictions and to the same order of approximation (viz., the single scattering hypothesis and that the extent of the distribution be $\ll r$)--i.e., that at the specular angle the effect of such a distribution may be found by multiplying the effect of the central boss (${}^{o}\psi_c$) by the total number of bosses.

2) Uniform Random Distribution.

Consider $\underline{E}^i$ incident on a uniform random distribution of parallel bosses of equal radii on an infinite yz-plane; the axes of the bosses being distributed along the y-axis according to a density function M which is independent of y, so that M dy equals the number of bosses in area of width dy by unit length (or essentially in dy). Analogous to the summation procedure of the preceding section and based as previously on the single scattering hypothesis we must now evaluate

$$\psi_c = M \int {}^{y}\psi_c \, e^{iK'y \sin\alpha} \, dy \qquad , \tag{23}$$

where the superscript y indicates that in (16) for ${}^{o}\psi_c$ we replace r, θ by r', θ' which are functions of y--i.e., the running coordinates from a point on the distribution to a fixed field point P (while r, θ, will be retained to specify the position of P in the reference coordinate frame at the center of the distribution).

(a) Small Finite Distribution .

For a finite distribution extending from $y = -d/2$ to $y = +d/2$ ($d \ll r$) we obtain by essentially the same procedure as for the grating (i.e., replacing $r'$ by $r - y \sin\theta$ and $\theta'$ by $\theta$, neglecting $y \sin\theta \ll r$ in the denominator, and evaluating the resulting simple integral as in reference 1,IIC1) that

$$(24) \qquad \psi_c = {}^{o}\psi_c \, Md \sin\Gamma/\Gamma$$

where $\Gamma$ is as in (19) with b replaced by d. Hence

$$(25) \qquad \underline{F}^r = \underline{F}^p + {}^{o}\underline{F}^c \, Md \sin\Gamma/\Gamma \qquad .$$

This may also be extended, by summation, to a superposition of such distributions which are parallel and coplanar and where each is characterized by a different density function, length and boss radius. Then provided that all $a_j = a$ we would again obtain expression (22) at the specular angle ($\Gamma_j = 0$, $\sin\Gamma_j/\Gamma_j \to 1$) with $\mathcal{M} = Md$ or $\mathcal{M} = \sum M_j d_j$ for the total number of bosses in the distribution.

(b) Infinite Distribution .

For a distribution extending from $y = -\infty$ to $y = +\infty$ we evaluate (27) at an arbitrary constant field point P, whose coordinates in the reference frame are $x = D = r\cos\theta$, $y = C = r\sin\theta$, $z = z$, by the method of stationary phase (section IIC2, reference 1). From (23) we obtain on substituting from (16)

$$(23') \qquad \psi_c' = B'e^{ihz}(\pi i/2K')^{\frac{1}{2}} (K'a)^2 \, 2\cos\alpha \, M\int_{-\infty}^{\infty} e^{iK'(r'-y\sin\alpha)}(r')^{-\frac{1}{2}}[\cos\theta'] \, dy \quad ,$$

$$\psi_c'' = -A'e^{ihz}(\pi i/2K')^{\frac{1}{2}}(K'\varepsilon)^2 M\int_{-\infty}^{\infty} e^{iK'(r'-y\sin\alpha)}(r')^{-\frac{1}{2}}[1+2\sin\alpha\sin\theta'] \, dy \quad .$$

The integrals may be expressed in terms of $\theta'$ as

$$(26) \qquad D^{\frac{1}{2}}\, e^{-iK'c\,\sin\alpha} \int_{-\pi/2}^{\pi/2} e^{iK'D(\sec\theta' + \tan\theta'\,\sin\alpha)} [\quad]\, \sec^{3/2}\theta'\, d\theta'$$

(where the bracket is either that of (23') or (23")) and yield on evaluation (the phase being stationary at $\theta' = -\alpha$)

$$(27) \qquad [\quad]_{\theta' = -\alpha}\; e^{iK'r\,\cos(\theta+\alpha)}\, \sec\alpha\, (2i\pi/K')^{\frac{1}{2}} \;, \qquad (\alpha \neq \pi/2) \quad .$$

Hence

$$(28) \qquad \psi'_c = iB'\, e^{i\underline{K}_p \cdot \underline{R}}\, K'M\pi a^2\; 2\cos^2\alpha\;\; \sec\alpha \;,$$

$$\psi''_c = -iA'\, e^{i\underline{K}_p \cdot \underline{R}}\, K'M\pi a^2\; \cos 2\alpha\; \sec\alpha \;;$$

and the scattered reflected potentials are

$$(29) \qquad \begin{aligned} \psi'_r &= \psi'_p + \psi'_c = \psi'_p\, (1 - iK'V4\,\cos\alpha) \;, \\ \psi''_r &= \psi''_p + \psi''_c = \psi''_p\, (1 - iK'V2\,\cos 2\alpha\,\sec\alpha) \;, \end{aligned}$$

where $V = M\pi a^2/2$ and $\psi'_p$, $\psi''_p$, are as in (10). The field as obtained from (15a,c) is then

$$\underline{E}^r = \underline{E}^p + \underline{E}^c \ ,$$

$$\begin{aligned}
\underline{E}^c = -\frac{i2K'V\, e^{i\underline{K}_p\cdot\underline{R}}}{\cos\alpha\,(1-\sin^2\beta\,\cos^2\alpha)^{\frac{1}{2}}}\Big\{ &\Big[\cos\beta\,(A\cos\beta\,\cos\alpha + B\sin\alpha)\cos(\theta+\alpha)2\cos^2\alpha \\
&+ (-B\cos\beta\,\cos\alpha + A\sin\alpha)\sin(\theta+\alpha)\cos2\alpha\Big]\hat{r} \\
&+\Big[-\cos\beta\,(A\cos\beta\,\cos\alpha + B\sin\alpha)\sin(\theta+\alpha)2\cos^2\alpha \\
&+ (-B\cos\beta\,\cos\alpha + A\sin\alpha)\cos(\theta+\alpha)\cos2\alpha\Big]\hat{\theta} \\
&-\sin\beta\,(A\cos\beta\,\cos\alpha + B\sin\alpha)2\cos^2\alpha\,\hat{k}\Big\}
\end{aligned}$$

(30)

$$= -\,i2K'V\,\sec\alpha\Big[\,2\cos^2\alpha\,\underline{E}'^p + \cos2\alpha\,\underline{E}''^p\Big] \qquad ;$$

$$\underline{H}^r = \underline{H}^p + \underline{H}^c \ ,$$

$$\begin{aligned}
\underline{H}^c = -\frac{\eta\, i2K'V\, e^{i\underline{K}_p\cdot\underline{R}}}{\cos\alpha\,(1-\sin^2\beta\,\cos^2\alpha)^{\frac{1}{2}}}\Big\{ &\Big[(A\cos\beta\,\cos\alpha + B\sin\alpha)\sin(\theta+\alpha)2\cos^2\alpha \\
&-\cos\beta\,(-B\cos\beta\,\cos\alpha + A\sin\alpha)\cos(\theta+\alpha)\cos2\alpha\Big]\hat{r} \\
&+\Big[(A\cos\beta\,\cos\alpha + B\sin\alpha)\cos(\theta+\alpha)2\cos^2\alpha \\
&+\cos\beta\,(-B\cos\beta\,\cos\alpha + A\sin\alpha)\sin(\theta+\alpha)\cos2\alpha\Big]\hat{\theta} \\
&+\sin\beta\,(-B\cos\beta\,\cos\alpha + A\sin\alpha)\cos2\alpha\,\hat{k}\Big\}
\end{aligned}$$

$$= -i2K'V\,\sec\alpha\Big[2\cos^2\alpha\,\underline{H}'^p + \cos2\alpha\,\underline{H}''^p\Big] \quad ,$$

where $\underline{E}^p$, $\underline{H}^p$, are as in (6).

For the case $\beta = \pi/2$ (incidence perpendicular to the axis of the boss) the above simplify considerably. Thus

$$\begin{aligned} \psi'_r &= -(B/K^2)\, e^{iKr\cos(\theta+\alpha)}\,(1 - iKV4\cos\alpha) \,; \\ \psi''_r &= \eta\,(A/K^2)\, e^{iKr\cos(\theta+\alpha)}\,(1 - iKV2\cos 2\alpha\sec\alpha) \end{aligned} \tag{29'}$$

(which except for the multiplying constants are identical with the velocity potentials for the cases of 0 and $\infty$ impedance--p.17, reference 1); and the field

$$\begin{aligned} \underline{E}^r &= \underline{E}^p + \underline{E}^c \,, \\ \underline{E}^c &= -i2KV\, e^{iKr\cos(\theta+\alpha)}\sec\alpha\left\{A\cos 2\alpha\left[\sin(\theta+\alpha)\hat{r} + \cos(\theta+\alpha)\hat{\theta}\right] - B2\cos^2\alpha\ \hat{k}\right\} \\ &= -i2KV\sec\alpha\left[\cos 2\alpha\ \underline{E}^p_{\parallel} + 2\cos^2\alpha\ \underline{E}^p_{\perp}\right] \,; \\ \underline{H}^r &= \underline{H}^p + \underline{H}^c \,, \\ \underline{H}^c &= -i\eta 2KV\, e^{iKr\cos(\theta+\alpha)}\sec\alpha\left\{B2\cos^2\alpha\left[\sin(\theta+\alpha)\hat{r} + \cos(\theta+\alpha)\hat{\theta}\right] + A\cos 2\alpha\ \hat{k}\right\} \\ &= -i2KV\sec\alpha\left[2\cos^2\alpha\ \underline{H}^p_{\parallel} + \cos 2\alpha\ \underline{H}^p_{\perp}\right] \,, \end{aligned} \tag{30'}$$

where $\underline{E}^p$ and $\underline{H}^p$ are as in (6') and where the subscripts $\parallel, \perp$, indicate the components $\parallel$ and $\perp$ to the plane of incidence. (Note the relationship of the c and p quantities of these expressions, from which it can be seen that were either A or B equal to zero then the scattered component (c) would be directly proportional to the specular component (p).

On adding the p and c terms we obtain

$$\begin{aligned} \underline{E}^r &= \left\{A(1 - i2KV\sec\alpha\cos 2\alpha)\left[\sin(\theta+\alpha)\hat{r} + \cos(\theta+\alpha)\hat{\theta}\right] - B(1 - i4KV\cos\alpha)\hat{k}\right\} e^{iKr\cos(\theta+\alpha)} \,, \\ H^r &= \eta\left\{B(1 - i4KV\cos\alpha)\left[\sin(\theta+\alpha)\hat{r} + \cos(\theta+\alpha)\hat{\theta}\right] + A(1 - i2KV\sec\alpha\cos 2\alpha)\hat{k}\right\} e^{iKr\cos(\theta+\alpha)} \,. \end{aligned}$$

There is little resemblance between the scattered components of these expressions and those of the finite distributions (20), (25) treated previously. The characteristic term of the single semicylindrical boss no longer appears; in fact, we have lost all trace of cylindrical waves and the scattered reflected wave consists of a superposition of plane waves of different amplitudes and phase traveling in the specular direction. The symbol V, the volumetric departure from the plane, is introduced to eliminate all trace of the boss from the expressions. Since $\pi a^2/2$ is the volume of a semicylinder of unit length, V represents the total volume of bosses per square centimeter of distribution and has the dimension of length.

For a superposition of such distributions, which are all parallel and coplanar, we obtain the above expressions containing $\sum_j V_j$. Hence, denoting $\sum_j V_j$, the total volumetric departure from the plane, by V yields the same expressions as for the single distribution.

We could of course have obtained (30) by performing the above operations (26), (28) directly on the field quantities as given by (17). The resulting expressions would then be in terms of the unit vectors associated with the stationary value rather than in terms of the reference unit vectors and would require transformation to facilitate comparison with the plane wave components. We will demonstrate the procedure for future reference.

We wish to evaluate

$$\underline{E}^c = M\int_{-\infty}^{\infty} {}^{y}\underline{E}^c \, e^{-iK'y\sin\alpha} \, dy \qquad (31)$$

For greater lucidity we will first perform the transformation and then the integration. For brevity we write

$${}^{y}\underline{E}^c = E_1\hat{r}' + E_2\hat{\theta}' + E_3\hat{k} \quad , \qquad (32)$$

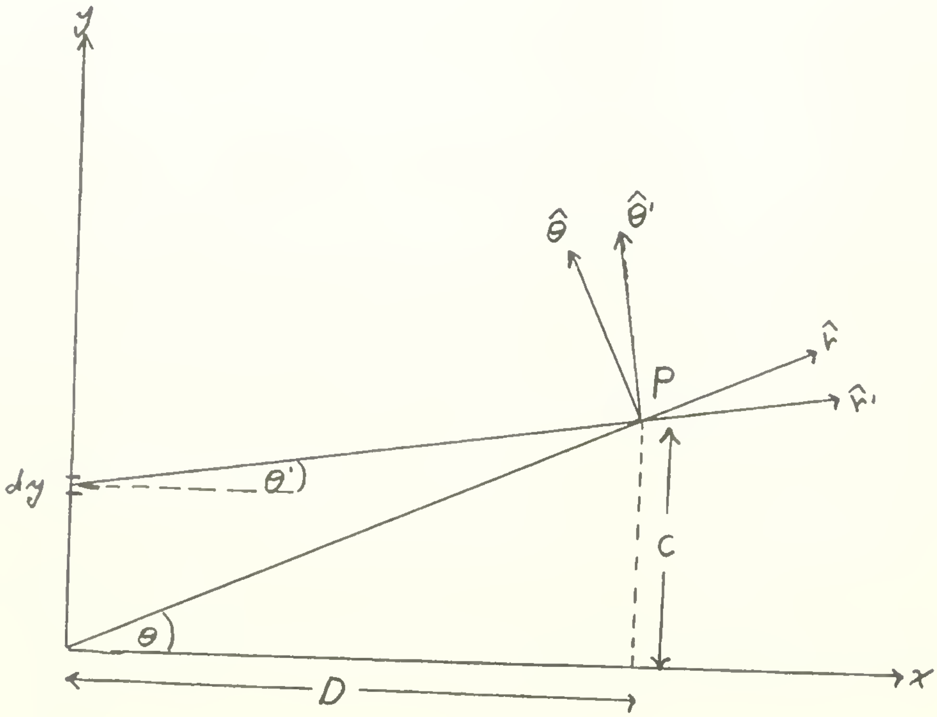
the scalar coefficients $E_1$, $E_2$, $E_3$ (whose values can be seen from comparison with (17)) being functions of the variables $r'$ and $\theta'$. We now transform this quantity and express it in terms of the base unit vectors of the reference frame; $\hat{r}$, $\hat{\theta}$, $\hat{k}$. Thus

$$
\begin{aligned}
{}^{y}\underline{E}^{c} &= {}^{y}\underline{E}^{c}\cdot\hat{r}\hat{r} + {}^{y}E^{c}\cdot\hat{\theta}\hat{\theta} + {}^{y}\underline{E}^{c}\cdot\hat{k}\hat{k} \\
&= (E_1\hat{r}'\cdot\hat{r} + E_2\hat{\theta}'\cdot\hat{r})\,\hat{r} + (E_1\hat{r}'\cdot\hat{\theta} + E_2\hat{\theta}'\cdot\hat{\theta})\hat{\theta} + E_3\hat{k} \qquad (33) \\
&= E_1'\hat{r} + E_2'\hat{\theta} + E_3'\hat{k} \quad .
\end{aligned}
$$

As may be seen from the sketch below, the scalar products are equal to

$$
(34) \qquad \hat{r}'\cdot\hat{r} = \cos(\theta - \theta') \quad , \quad \hat{\theta}'\cdot\hat{r} = \cos[\pi/2 - (\theta - \theta')] = \sin(\theta - \theta') \ ,
$$

$$
\hat{r}'\cdot\hat{\theta} = \cos[\pi/2 + (\theta - \theta')] = -\sin(\theta - \theta') \quad , \quad \hat{\theta}'\cdot\hat{\theta} = \cos(\theta - \theta') \quad .
$$



Consequently

$$ {}^{y}\underline{E}^{c} = \left[E_1\cos(\theta - \theta') + E_2\sin(\theta - \theta')\right]\hat{r} + \left[-E_1\sin(\theta - \theta') + E_2\cos(\theta - \theta')\right]\hat{\theta} + E_3\hat{k} \quad . \tag{35} $$

Expression (31) is then equal to

$$ \begin{aligned} \underline{E}^{c} = -\frac{e^{ihz}}{(1-\sin^2\beta\ \cos^2\alpha\ )^{\frac{1}{2}}}\ (\pi i/2K')^{\frac{1}{2}}(K'a)^2 M \int_{-\infty}^{\infty} e^{iK'(r'-y\ \sin\alpha)}(r')^{-1/2} \\ \times\Big\{\Big[\cos\beta\ (A\cos\beta\ \cos\alpha + B\ \sin\alpha\ )2\ \cos\alpha\cos\ \theta\ \cos\ (\theta - \theta') \\ + (-B\ \cos\beta\cos\alpha + A\ \sin\alpha\ )(1 + 2\ \sin\alpha\sin\ \theta')\ \sin\ (\theta - \theta')\Big]\ \hat{r} \\ + \Big[-\cos\beta\ (A\ \cos\beta\ \cos\alpha + B\ \sin\alpha\ )2\ \cos\alpha\ \cos\ \theta'\ \sin\ (\theta - \theta') \\ + (-B\ \cos\beta\ \cos\alpha + A\ \sin\alpha\ )(1 + 2\ \sin\alpha\sin\ \theta')\ \cos\ (\theta - \theta')\Big]\ \hat{\theta} \\ - \sin\beta\ (A\ \cos\beta\ \cos\alpha + B\ \sin\alpha\ )2\ \cos\alpha\ \cos\ \theta'\ \hat{k}\Big\}\ dy \end{aligned} $$

which on evaluation, as previously, yields (30).

## C. Distributions of Cylinders .

The results of the preceding section can be readily extended to the problems of $\underline{E}^{i}$ (as in (5)) incident on similar distributions of whole cylinders. The expressions for the $\psi_c$ and $\underline{F}^{c}$ for the small finite distributions (as in (19)-(25)) are formally identical with those that would be obtained for the analogous distributions of cylinders if it is understood that ${}^{o}\psi_c$ and ${}^{o}\underline{F}^{c}$ represent the approximate solutions for the single cylinder at the center of the distribution for $K'r \gg 1$, $K'a \ll 1$.

The solutions for the single cylinder may be readily obtained by what is essentially a simplification of the previous procedure. Thus from (10) we obtain the forms of the $\psi_i$ and consider them to be incident on the cylinder. The total potentials are then

$$\psi = \psi_i + \psi_c = C\, e^{ihz} \sum i^n \varepsilon_n \left[ J_n + A_n H_n \right] \cos n(\pi - \theta + \alpha) \quad ;$$

where for $\psi'$, $C = B'$, $A_n = A'_n$; while for $\psi''$, $C = A'$, $A_n = A''_n$. The scattered potentials, $\psi_c$, are then equal to

$$\psi'_c = B'\, e^{ihz} \sum i^n \varepsilon_n A'_n H_n \cos n\,(\pi - \theta + \alpha) \quad ,$$

$$\psi''_c = A'\, e^{ihz} \sum i^n \varepsilon_n A''_n H_n \cos n\,(\pi - \theta + \alpha) \quad ,$$

where the $A'_n$, $A''_n$, are as in (13). The rest follows as previously and to the order of $(K'a)^2/(K'r)^{\frac{1}{2}}$ we obtain

$$\begin{aligned} {}^0\psi'_c &= B'\, e^{ihz + iK'r} (i\pi/2K'r)^{\frac{1}{2}} \left[ 1/\ln(\gamma K'a/2) + (K'a)^2 \cos(\theta - \alpha) \right] , \\ {}^0\psi''_c &= -A'\, e^{ihz + iK'r} (i\pi/2K'r)^{\frac{1}{2}} (K'a)^2 \left[ 1 + 2\cos(\theta - \alpha) \right]/2 \quad ; \end{aligned} \tag{37}$$

and the field as obtained from (15a,c)

$$\begin{aligned} {}^0\underline{E}^c = &- \frac{e^{ihz + iK'r}}{(1 - \sin^2\beta \cos^2\alpha)^{\frac{1}{2}}} (i\pi/2K'r)^{\frac{1}{2}} \\ &\times \Big\{ \cos\beta\, (A\cos\beta \cos\alpha + B\sin\alpha) \left[ 1/\ln(\gamma K'a/2) + (K'a)^2 \cos(\theta - \alpha) \right] \hat{r} \\ &+ (-B\cos\beta \cos\alpha + A\sin\alpha)(K'a)^2 (\left[ 1 + 2\cos(\theta - \alpha) \right]/2)\, \hat{\theta} \\ &- \sin\beta\, (A\cos\beta \cos\alpha + B\sin\alpha) \left[ 1/\ln(\gamma K'a/2) + (K'a)^2 \cos(\theta - \alpha) \right] \hat{k} \Big\} , \end{aligned} \tag{38}$$

$${}^0\underline{H}^c = -\eta \left\{ \cos\beta\, {}^0E^c_\theta\, \hat{r} + ({}^0E^c_z/\sin\beta)\, \hat{\theta} - \sin\beta\, {}^0E^c_\theta\, \hat{k} \right\} \quad .$$

Had the incident wave been $\underline{E}^p$ (as in (6)) the above would be modified only to the extent that $\cos(\theta - \alpha)$ would be replaced by $-\cos(\theta + \alpha)$ and that the bracket containing the ln term (that of ${}^{o}\psi'_c$) would be multiplied by -1. (It should be noted that the $(K'a)^2$ term in the bracket containing the logarithm is included only to facilitate comparison with the previously derived expressions for the bosses and that its contribution is negligible compared to the logarithmic term). It can then readily be seen that if these two cases are added (i.e., $\underline{E}^i$ and $\underline{E}^p$ simultaneously incident on the cylinder, the identical image problem treated for the bosses) the ln terms cancel while we have $\cos(\theta - \alpha) + \cos(\theta + \alpha) = 2 \cos\theta \cos\alpha$ for $\psi'_c$, and we have for $\psi''_c$ that $[1 + 2\cos(\theta - \alpha) + 1 - 2\cos(\theta + \alpha)]/2 = 1 + 2 \sin\theta \sin\alpha$, as was of course expected. (See expressions (16)).

For the infinite distribution and an incident wave $\underline{E}^i$, we obtain by the method of stationary phase applied to (23) utilizing the potentials of (37), that

$$\psi'_c = iB' e^{i\underline{K}_q \cdot \underline{R}} K' M \pi a^2 \sec\alpha \, Q' ,$$

(39)

$$\psi''_c = iA' e^{i\underline{K}_q \cdot \underline{R}} K' M \pi a^2 \sec\alpha \, Q'' ;$$

where for the reflected wave

(40) $q = p,\ Q' = 1/(K'a)^2 \ln(\gamma K'a/2) + \cos 2\alpha ,\ Q'' = [1 + 2\cos 2\alpha]/2$ ,

while for the transmitted wave (obtained by replacing $\theta'$ by $-\theta'$ before evaluating the integral)

(41) $q = i,\ Q' = [1/(K'a)^2 \ln(\gamma K'a/2) - 1] ,\ Q'' = -1/2$

Had the incident wave been $\underline{E}^p$ rather than $\underline{E}^i$ the above would be altered only to the extent of interchanging i and p in q and multiplying Q' by -1. As previously the addition of these two cases may be used as a check on the results derived for the bosses. Thus for $x > 0$ ($q = p$, the reflected wave for $\underline{E}^i$ and the transmitted wave for $\underline{E}^p$) the ln terms cancel to yield $\cos 2\alpha + 1 = 2\cos^2\alpha$ for $\psi_c'$ while $[(1+2\cos 2\alpha)/2] - 1/2 = \cos 2\alpha$ for $\psi_c''$ as obtained previously. (See expressions (28)).

The corresponding field quantities may be readily obtained from the above by differentiation as in (15a,c), or by inspection of (30). Thus, $\underline{F}^c = iK'M\pi a^2(Q'\underline{F}'^q + Q''\underline{F}''^q)$.

(It should again be noted that, in the ln bracket, it is the ln term that predominates and the other term which is negligible is included only to facilitate comparison. It should also be noted that $M\pi a^2$ is the volume density of cylinders or essentially the volume of material per centimeter of distribution along the y-axis).

## 4. The Hemispherical Boss .

### A. The Single Boss .

Consider a plane electromagnetic plane wave characterized by $\underline{E}^i$ incident on a hemispherical boss of radius-a on an infinite xy-plane as in Fig. 2a. Without loss of generality the plane of incidence is taken to be the xz-plane. We seek a solution of the problem as formulated in section 2 subject to the boundary condition that the tangential component of E vanishes on the plane and boss, or that

$$(42) \qquad (E_\varphi = E_r = 0, \text{ or } E_y = E_x = 0)_{\theta = \pi/2,\ z = 0}$$

$$(E_\varphi = E_\theta = 0)_{r = a} \quad .$$

Fig. 2(a)

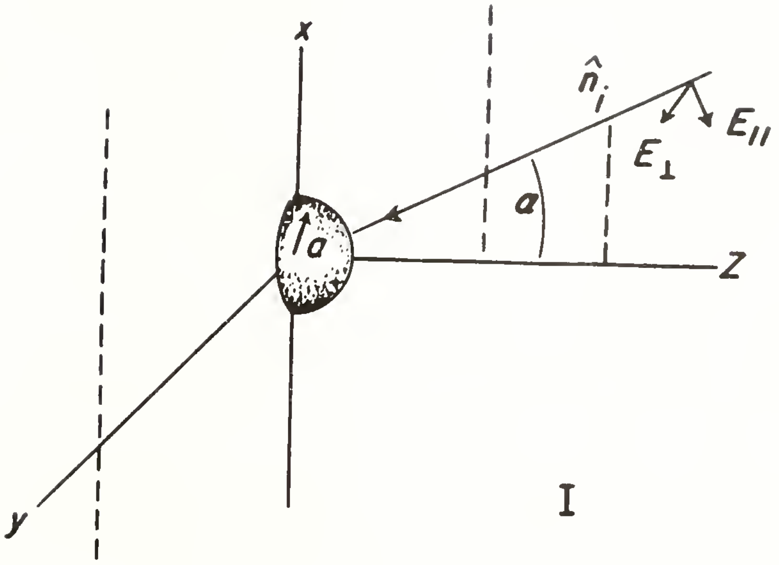


Fig. 2(b)

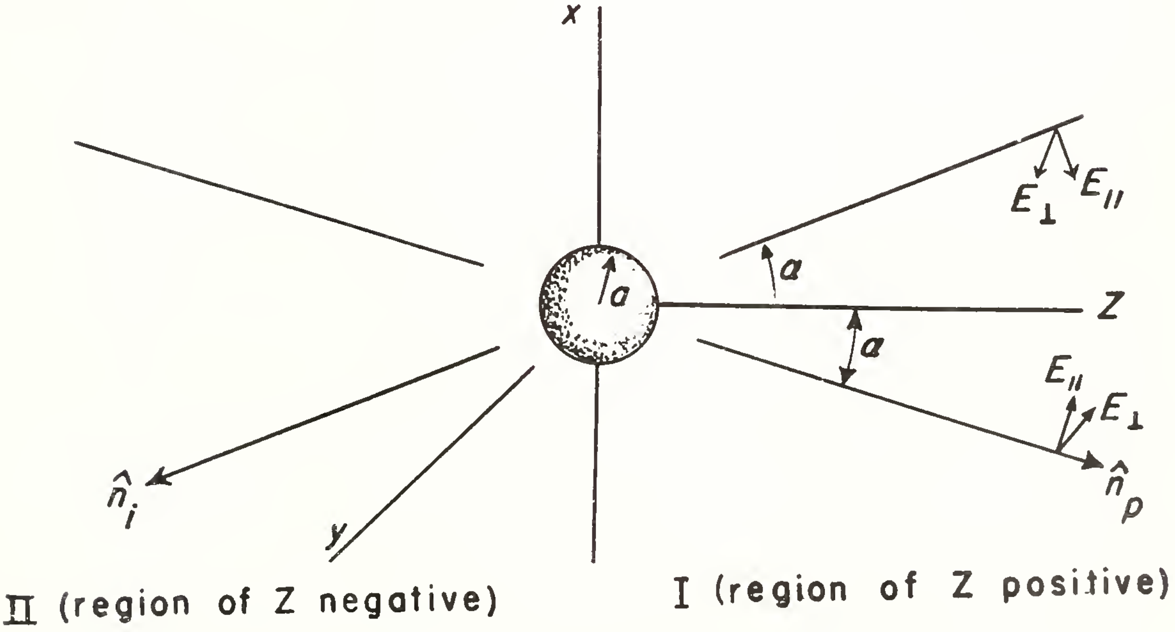


Figure 2

(a) A plane wave incident on a hemispherical boss on an infinite xy-plane. The plane of incidence is the xz-plane in which $\hat{n}_i$ makes an angle $\alpha$ with the z-axis.

(b) Two plane waves, $\hat{n}_i$ and its image in the xy-plane, $\hat{n}_p$, simultaneously incident on a sphere at the origin.

The image thechnique discussed and employed previously will be employed and we will concern ourselves with the problem as represented by Fig. 2b and seek to determine the Hertz potentials of the form $\Pi = \psi\underline{r}$.

In general the incident field can be initially decomposed into components parallel and perpendicular to the xz-plane of incidence, expressed in terms of the reference cartesian coordinates and then transformed to the spherical coordinates of the boss, as follows sequentially:

$$\begin{aligned}\underline{E}^i &= A\underline{E}^i_{\parallel} + B\underline{E}^i_{\perp} = \{A(-\cos\alpha\,\hat{i} + \sin\alpha\,\hat{k}) + B\hat{j}\}\, e^{i\underline{K}_i\cdot\underline{r}} \\ &= e^{i\underline{K}_i\cdot\underline{r}}\, \{[A(-\cos\alpha\,\sin\theta\,\cos\varphi + \sin\alpha\,\cos\theta) + B\,\sin\theta\,\sin\varphi]\hat{r} \\ &\qquad + [-A(\cos\alpha\,\cos\theta\,\cos\varphi + \sin\alpha\,\sin\theta) + B\,\cos\theta\,\sin\varphi]\hat{\theta} \\ &\qquad + [A\,\cos\alpha\,\sin\varphi + B\,\cos\varphi]\hat{\varphi}\} \quad ,\end{aligned} \tag{43}$$

and

$$\begin{aligned}\underline{H}^i &= \eta\,\hat{n}_i \times \underline{E}^i = \eta\{B(\cos\alpha\,\hat{i} - \sin\alpha\,\hat{k}) + A\hat{j}\}\, e^{i\underline{K}_i\cdot\underline{r}} \\ &= \eta\, e^{i\underline{K}_i\cdot\underline{r}}\, \{[B(\cos\alpha\,\sin\theta\,\cos\varphi - \sin\alpha\,\cos\theta) + A\,\sin\theta\,\sin\varphi]\hat{r} \\ &\qquad + [B(\cos\alpha\,\cos\theta\,\cos\varphi + \sin\alpha\,\sin\theta) + A\,\cos\theta\,\sin\varphi]\hat{\theta} \\ &\qquad + [-B\,\cos\alpha\,\sin\varphi + A\,\cos\varphi]\hat{\varphi}\} \quad .\end{aligned}$$

In these expressions the position vector of a field point P is $\underline{r} = r(\sin\theta\,\cos\varphi\,\hat{i} + \sin\theta\,\sin\varphi\,\hat{j} + \cos\theta\,\hat{k})$; the incident propagation vector is

$$\underline{K}_i = K\hat{n}_i = -K(\sin\alpha\,\hat{i} + \cos\alpha\,\hat{k}) \qquad ;$$

and the transformations

$$\begin{aligned}\hat{i} &= \sin\theta\cos\varphi\,\hat{r} + \cos\theta\cos\varphi\,\hat{\theta} - \sin\varphi\,\hat{\varphi}\,,\\ \hat{j} &= \sin\theta\sin\varphi\,\hat{r} + \cos\theta\sin\varphi\,\hat{\theta} + \cos\varphi\,\hat{\varphi}\,,\\ \hat{k} &= \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\,,\end{aligned}$$

have been employed. It can be seen that $\underline{H}^i$ may be obtained from $\underline{E}^i$ by replacing A by -B and B by A and multiplying the resulting expression by $\eta$.

The image wave in the xy-plane (characterized by $\underline{E}^p$ and $\underline{H}^p$) is the specularly reflected wave that would be obtained were the boss absent and is determined by imposing the first of conditions (42) on $\underline{E}^i + \underline{E}^p$. Thus on replacing $\underline{K}_i$ by the image propagation vector $\underline{K}_p = K\hat{n}_p = K(-\sin\alpha\,\hat{i} + \cos\alpha\,\hat{k})$ we obtain $\underline{E}^p$ by changing the signs of the $\hat{i}$ and $\hat{j}$ components of $\underline{E}^i$, and $\underline{H}^p$ by changing the sign of the $\hat{k}$ component of $\underline{H}^i$. Hence we obtain

$$\begin{aligned}\underline{E}^p &= \left\{A(\cos\alpha\,\hat{i} + \sin\alpha\,\hat{k}) - B\hat{j}\right\} e^{i\underline{K}_p\cdot\underline{r}}\\ &= e^{i\underline{K}_p\cdot\underline{r}}\Big\{\left[A(\cos\alpha\sin\theta\cos\varphi + \sin\alpha\cos\theta) - B\sin\theta\sin\varphi\right]\hat{r}\\ &\qquad + \left[A(\cos\alpha\cos\theta\cos\varphi - \sin\alpha\sin\theta) - B\cos\theta\sin\varphi\right]\hat{\theta}\\ &\qquad - \left[A\cos\alpha\sin\varphi + B\cos\varphi\right]\hat{\varphi}\Big\}\,,\end{aligned}\tag{44}$$

and

$$\begin{aligned}\underline{H}^p &= \eta\,\hat{n}_p \times \underline{E}^p = \eta\left\{B(\cos\alpha\,\hat{i} + \sin\alpha\,\hat{k}) + A\hat{j}\right\} e^{i\underline{K}_p\cdot\underline{r}}\\ &= \eta\, e^{i\underline{K}_p\cdot\underline{r}}\Big\{\left[B(\cos\alpha\sin\theta\cos\varphi + \sin\alpha\cos\theta) + A\sin\theta\sin\varphi\right]\hat{r}\\ &\qquad + \left[B(\cos\alpha\cos\theta\cos\varphi - \sin\alpha\sin\theta) + A\cos\theta\sin\varphi\right]\hat{\theta}\\ &\qquad + \left[-B\cos\alpha\sin\varphi + A\cos\varphi\right]\hat{\varphi}\Big\}\,.\end{aligned}$$

To proceed further and discover the field $\underline{E}^s$ and $\underline{H}^s$ scattered by the sphere subject to the boundary condition at $r = a$, we now find it convenient to introduce two Hertz potentials of the form $II = \psi \underline{r}$. Substituting into (3) yields

$$\nabla \times \nabla \times (\psi \underline{r}) - \nabla\nabla\cdot(\psi\underline{r}) - K^2\psi\underline{r} = 0$$

$$= \nabla \times (\nabla\psi \times \underline{r} + \cancel{\psi\nabla\times\underline{r}}) - \nabla(\underline{r}\cdot\nabla\psi) - \nabla(\psi\nabla\cdot\underline{r}) - K^2\psi\underline{r} = 0$$

$$= (\nabla\psi)\nabla\cdot\underline{r} - \underline{r}\nabla^2\psi + (\underline{r}\cdot\nabla)\nabla\psi - (\nabla\psi\cdot\nabla)\underline{r} - (\underline{r}\cdot\nabla)\nabla\psi + (\nabla\psi\cdot\nabla)\underline{r} + \nabla\psi\times\cancel{(\nabla\times\underline{r})}$$

$$+ \underline{r}\times\cancel{(\nabla\times\nabla\psi)} - (\nabla\psi)\nabla\cdot\underline{r} - \psi\nabla\nabla\cdot\underline{r} - K^2\psi\underline{r} = 0 \rightarrow$$

$$= (\nabla^2 + K^2)\psi = 0$$

so that the $\psi$ are the usual spherical eigenfunctions.[10] We therefore obtain from (1) and (2) that the field can be expressed as the sum of the two partial fields

$$\underline{E}' = (K^2 + \nabla\nabla\cdot)(\psi'\underline{r}) = K^2\psi'\underline{r} + \nabla[(\psi' r)\nabla\cdot\hat{r} + \hat{r}\cdot\nabla(\psi' r) = K^2\psi'\underline{r} + \nabla\left[\frac{\partial(\psi' r)}{\partial r}\right]$$

$$= \left\{\hat{r}\left(K^2 + \frac{\partial^2}{\partial r^2}\right) + \hat{\theta}\,\frac{1}{r}\,\frac{\partial^2}{\partial r\,\partial\theta} + \hat{\varphi}\,\frac{1}{r\sin\theta}\,\frac{\partial^2}{\partial r\,\partial\varphi}\right\}(\psi' r),$$

(45)

$$\underline{H}' = -iK\eta\nabla\times(\psi'\underline{r}) = iK\eta\,\hat{r}\times\nabla(\psi' r)$$

$$= iK\eta\left\{-\hat{\theta}\,\frac{1}{r\sin\theta}\,\frac{\partial}{\partial\varphi} + \hat{\varphi}\,\frac{1}{r}\,\frac{\partial}{\partial\theta}\right\}(\psi' r) \quad ;$$

and

$$\underline{E}'' = i\mu\omega\nabla\times(\psi''\underline{r}) = i\mu\omega\left\{\hat{\theta}\,\frac{1}{r\sin\theta}\,\frac{\partial}{\partial\varphi} - \hat{\varphi}\,\frac{1}{r}\,\frac{\partial}{\partial\theta}\right\}(\psi'' r),$$

(46)

$$\underline{H}'' = (K^2 + \nabla\nabla\cdot)(\psi''\underline{r}) = \left\{\hat{r}\left(K^2 + \frac{\partial^2}{\partial r^2}\right) + \hat{\theta}\,\frac{1}{r}\,\frac{\partial^2}{\partial r\,\partial\theta} + \hat{\varphi}\,\frac{1}{r\sin\theta}\,\frac{\partial^2}{\partial r\,\partial\varphi}\right\}(\psi'' r);$$

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[10] A discussion of the functions and expansions used in this section will be found in Stratton (reference 5) Ch. VII.

where

$$\underline{E}' = \underline{E}'^{i} + \underline{E}'^{p} + \underline{E}'^{s} \quad , \qquad \underline{E}'' = \underline{E}''^{i} + \underline{E}''^{p} + \underline{E}''^{s} \; ,$$

$$\underline{H}' = \underline{H}'^{i} + \underline{H}'^{p} + \underline{H}'^{s} \quad , \qquad \underline{H}'' = \underline{H}''^{i} + \underline{H}''^{p} + \underline{H}''^{s} \; ,$$

$$\psi' = \psi'_i + \psi'_p + \psi'_s \quad , \qquad \psi'' = \psi''_i + \psi''_p + \psi''_s \; ,$$

will now be determined. It should be noted from (45) and (46) that $E_r = E'_r = (K^2 + D_r^2)(\psi' r)$ and $H_r = H''_r = (K^2 + D_r^2)(\psi'' r)$ (where $D_r^2 = \frac{\partial^2}{\partial r^2}$, for brevity). The following procedure hinges on these two expressions since the radial components of the incident and image waves yield the corresponding $\psi$'s with which we determine the scattered $\psi$'s and hence the scattered field. It should also be noted that the primed quantities (of the electric type) have no component of H in the $\underline{r}$ direction and are therefore the TM mode, while the double primed quantities (of the magnetic type) have no component of E parallel to $\underline{r}$ and are therefore the TE mode--transverse referring of course to the vectorial direction of II.

The procedure from this point on is slightly more complicated than for the analogous cylindrical problem.[11] We determine the associated Hertz potentials from the radial components of (43) and (44). Now

$$e^{i\underline{K}_1 \cdot \underline{r}} = e^{-iKr(\sin\theta\cos\varphi\sin\alpha + \cos\theta\cos\alpha)}$$

$$= \sum_{n=0}^{\infty} i^n (2n+1) j_n(Kr) \sum_{m=0}^{n} \varepsilon_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) P_n^m(\cos\alpha) \cos m\varphi \, (-1)^n$$

$$\equiv \sum \mathrm{II}_{nm} \, (-1)^n \quad ,$$

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[11] The treatment of the problem of a plane electromagnetic wave incident on a sphere on which the essentials of the following development is based will be found in either Frank-Mises (reference 8) pp. 871-875 or M. Born, Optik, pp. 274-285, Edwards, Michigan, 1943.

(where the $j_n$ are the spherical Bessel functions in Morse's notation and the $P_n^m$ are the associated Legendre functions) while

$$e^{i\underline{K}_p\cdot\underline{r}} = e^{iKr(-\sin\theta\cos\varphi\sin\alpha+\cos\theta\cos\alpha)} = \sum II_{nm}\,(-1)^m.$$

(The symbols $II_{nm}$ above and the host that follow are introduced for brevity and will be discarded as soon as they have served their purpose). Consequently we can formally express the radial components as

$$E_r^i = \left(A\frac{\partial}{\partial\alpha} + \frac{B}{\sin\alpha}\frac{\partial}{\partial\varphi}\right)\frac{e^{i\underline{K}_i\cdot\underline{r}}}{iKr}$$

$$= \frac{1}{iKr}\sum_{n=1}^{\infty} i^n(2n+1)j_n\sum_{m=0}^{n}\mathcal{E}_m\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)\left\{AP_n^m(\cos\alpha)_\alpha\cos m\varphi\right.$$

$$\left.-\frac{B}{\sin\alpha}P_n^m(\cos\alpha)\,m\sin m\varphi\right\}(-1)^n \equiv \sum G'_{nm}\,(-1)^n \quad ,$$

(48)

$$E_r^p = -\left(A\frac{\partial}{\partial\alpha} + \frac{B}{\sin\alpha}\frac{\partial}{\partial\varphi}\right)\frac{e^{i\underline{K}_p\cdot\underline{r}}}{iKr} = -\sum G'_{nm}\,(-1)^m \quad ,$$

$$H_r^i = \eta\left(-B\frac{\partial}{\partial\alpha} + \frac{A}{\sin\alpha}\frac{\partial}{\partial\varphi}\right)\frac{e^{i\underline{K}_i\cdot\underline{r}}}{iKr}$$

$$= \frac{\eta}{iKr}\sum_{n=1}^{\infty} i^n(2n+1)\,j_n\sum_{m=0}^{n}\mathcal{E}_m\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)\left\{-BP_n^m(\cos\alpha)_\alpha\cos m\varphi\right.$$

$$\left.-\frac{A}{\sin\alpha}P_n^m(\cos\alpha)\,m\sin m\varphi\right\}(-1)^n = \sum G''_{nm}\,(-1)^n \quad ,$$

$$H_r^p = \eta\left(-B\frac{\partial}{\partial\alpha} + \frac{A}{\sin\alpha}\frac{\partial}{\partial\varphi}\right)\frac{e^{i\underline{K}_p\cdot\underline{r}}}{iKr} = \sum G''_{nm}\,(-1)^m$$

(where the subscript $\alpha$ indicates differentiation with respect to $\alpha$ , i.e.

$$P_n^m(\cos\alpha)_\alpha = \frac{\partial}{\partial\alpha}P_n^m(\cos\alpha) \qquad ).$$

Now from (45) and (46) we have

$$E_r^i = (K^2 + D_r^2)(\psi'_i r) \quad ,$$

$$E_r^p = (K^2 + D_r^2)(\psi'_p r) \quad ,$$

(49)

$$H_r^i = (K^2 + D_r^2)(\psi''_i r) \quad ,$$

$$H_r^p = (K^2 + D_r^2)(\psi''_p r) \quad .$$

It therefore seems feasible (from comparison of (48) and (49)) to seek solutions of (49) in the form

$$\psi'_i = \sum \alpha_n G'_{nm} (-1)^n \quad ,$$

$$\psi'_p = - \sum \alpha_n G'_{nm} (-1)^m \quad ,$$

(50)

$$\psi''_i = \sum \alpha_n G''_{nm} (-1)^n \quad ,$$

$$\psi''_p = \sum \alpha_n G''_{nm} (-1)^m \quad .$$

Substituting the quantities on the extreme right of (48) into the left sides of (49) and the $\psi$'s of (50) into the right sides of (49) yields a formal identity which reduces to a relation involving only the radial functions. Thus

(51) $$\sum G_{nm} = (K^2 + D_r^2)(r \sum \alpha_n G_{nm}) \rightarrow j_n = (K^2 + D_r^2)(r j_n \alpha_n)$$

which is valid for all four quantities. But from the spherical Bessel's equation (viz., the equation of the radial function resulting from the separation of the variables in the wave equation in spherical coordinates) we have

(52) $$(K^2 + D_r^2)(j_n r) = n(n+1) j_n / r \quad .$$

We therefore obtain from comparison of (51) and (52) that

$$\alpha_n = r/n(n+1) \quad . \tag{53}$$

Hence the Hertz potentials may be expressed as

$$\psi_i' = \frac{1}{K} \sum_{n=1}^{\infty} i^{n-1} \frac{(2n+1)}{n(n+1)} j_n \sum_{m=o}^{n} \varepsilon_m \frac{(n-m)!}{(n+m)!} P_n^m (\cos\theta)(-1)^n$$

$$\times \left\{ AP_n^m (\cos\alpha)_\alpha \cos m\varphi - \frac{B}{\sin\alpha} P_n^m (\cos\alpha)\, m \sin m\varphi \right\} = \sum j_n P'_{nm} (-1)^n \ ,$$

$$\psi_p' = - \sum j_n P'_{nm} (-1)^m \quad ,$$

(54)

$$\psi_i'' = -\frac{\eta}{K} \sum_{n=1}^{\infty} i^{n-1} \frac{(2n+1)}{n(n+1)} j_n \sum_{m=o}^{n} \varepsilon_m \frac{(n-m)!}{(n+m)!} P_n^m (\cos\theta)(-1)^n$$

$$\times \left\{ BP_n^m (\cos\alpha)_\alpha \cos m\varphi + \frac{A}{\sin\alpha} P_n^m (\cos\alpha) m \sin m\varphi \right\} = \sum j_n P''_{nm} (-1)^n \ ,$$

$$\psi_p'' = \sum j_n P''_{nm} (-1)^m \quad .$$

We are now in a position to solve the problem of Fig. 2b subject to the second condition of (42): i.e., we consider the original wave and its image simultaneously incident on the sphere and seek to determine the scattered field $\underline{E}^s$, $\underline{H}^s$, by first deriving the corresponding $\psi_s$.

We have

$$\psi_i' + \psi_p' = \sum j_n P'_{nm} \left[ (-1)^n - (-1)^m \right] = 2 \sum_n \sum_m (-1)^n j_n P'_{nm} \ ,$$

(55)

$$\psi_i'' + \psi_p'' = \sum j_n P''_{nm} \left[ (-1)^n + (-1)^m \right] = 2 \sum_n \sum_m (-1)^n j_n P''_{nm} \ ,$$

(where the notation $\sum_n \reflectbox{$\sum$}_m$ indicates a sum over n and m so that the indices have odd parity while $\sum_n \sum_m$ indicates that the indices have even parity). When these plane waves are incident on the sphere the corresponding scattered waves may be expressed in the same form as (55) but with $j_n$ replaced by $a_n h_n$, where the $h_n$, the spherical Hankel functions of the first kind, are associated with outgoing spherical waves. We therefore write

$$\psi' = \psi'_i + \psi'_p + \psi'_s = 2 \sum \reflectbox{$\sum$} (-1)^n \, (j_n + a'_n h_n) \; P'_{nm} \quad ,$$

(56)

$$\psi'' = \psi''_i + \psi''_p + \psi''_s = 2 \sum \sum (-1)^n \, (j_n + a''_n h_n) \; P''_{nm} \quad .$$

The constants $a'_n$, $a''_n$, are now chosen to satisfy the boundary condition $(E_\varphi = E_\theta = 0)_{r=a}$. From (45) and (46) it can be seen that as far as the dependence on r is concerned we may write $E'_\varphi, E'_\theta \propto \frac{1}{r} D_r \, (\psi'_r)$ while $E''_\varphi, E''_\theta \propto \psi''$, so that the restrictions on the $\psi$'s are that $\left[ D_r(\psi' r) = \psi'' = 0 \right]_{r=a}$. Imposing these on (56) (i.e., replacing r by a in $D_r(\psi' r)$ and in $\psi''$ and then equating the coefficients of the angular functions to zero) yields

$$(57) \qquad a'_n = -\left[ Kaj_n \; (Ka) \right]' \Big/ \left[ Kah_n \; (Ka) \right]' \quad , \; a''_n = - \, j_n(ka)/h_n(Ka) \quad ,$$

where the prime on the square bracket indicates differentiation with respect to the argument, Ka. (Contrast these with 13).

Consequently (56) and the field derivable by means of (45) and (46) represent the solution to the image problem of Fig. 2b, and it can be seen that this is also the solution to the boss problem in the region $\theta \leqslant \pi/2$ satisfying the required boundary conditions at r = a and $\theta = \pi/2$. The latter is of course obvious from (56) where it can be seen that $\psi_s$ has the same angular dependence as $\psi_i + \psi_p$ and must therefore satisfy the same boundary condition at $\theta = \pi/2$ since this condition depends only on $\theta$ or derivatives with respect to $\theta$.

To see analytically that the condition $(E_\varphi = E_r = 0)_{\theta=\pi/2}$ is satisfied, we note from (45) and (46) that (as far as the dependence on $\theta$ is concerned) $E'_r$ and $E'_\varphi \propto \Psi'$ while $E''_r = 0$, $E''_\varphi \propto D_\theta \Psi''$. Hence $E'_r$ and $E'_\varphi \propto P^{m_o^e}_{n_e^o}(\cos\theta)$ while $E''_\varphi \propto \frac{\partial}{\partial\theta} P^{m_e^o}_{n_e^o}(\cos\theta)$, so that all three quantities are zero at $\theta = \pi/2$ as is required.

Hence when a plane electromagnetic wave characterized by $\underline{E}^1$ as in (43) is incident on the boss on the infinite plane the scattered reflected Hertz potentials are

$$\Psi'_r = \Psi'_p + \Psi'_s = \Psi'_p + \frac{2}{K}\sum_{n=1}^{\infty} i^{-n-1}\frac{(2n+1)}{n(n+1)} a'_n h_n \sum_{m=0}^{n} \varepsilon_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta)$$

$$(58) \qquad \times\left\{ A P_n^m(\cos\alpha)_\alpha \cos m\varphi - B\frac{P_n^m(\cos\alpha)}{\sin\alpha} m \sin m\varphi \right\},$$

$$\Psi''_r = \Psi''_p + \Psi''_s = \Psi''_p - \frac{2\eta}{K}\sum_{n=1}^{\infty} i^{-n-1}\frac{(2n+1)}{n(n+1)} a''_n h_n \sum_{m=0}^{n} \varepsilon_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta)$$

$$\times\left\{ B P_n^m(\cos\alpha)_\alpha \cos m\varphi + A\frac{P_n^m(\cos\alpha)}{\sin\alpha} m \sin m\varphi \right\}.$$

The scattered reflected electromagnetic field is then

$$(59) \qquad \underline{E}^r = \underline{E}^p + \underline{E}^s ,$$

where by means of equations (45) and (46)

$$\underline{E}^s = \underline{E}'^s + \underline{E}''^s$$

$$(59a) \qquad = \left[K^2 + \frac{\partial^2}{\partial r^2}\right](\Psi'_s r)\hat{r} + \left[\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}(\Psi'_s r) + \frac{i\mu\omega}{\sin\theta}\frac{\partial\Psi''_s}{\partial\varphi}\right]\hat{\theta} + \left[\frac{1}{r\sin\theta}\frac{\partial^2}{\partial r\partial\varphi}(\Psi'_s r) - i\mu\omega\frac{\partial\Psi''_s}{\partial\theta}\right]\hat{\varphi}.$$

Making use of (58) we obtain

$$\underline{E}^s = \frac{1}{Kr} \sum_m N n(n+1)\, a_n' h_n \sum_m M\, P(t) \left[ A\, P(a)_\alpha C - B \frac{P(a)}{\sin\alpha} m S \right] \hat{r}$$

$$+ \sum_n N \left\{ a_n' \frac{[Krh_n]'}{Kr} \sum_m M\, P(t)_\theta \left[ A\, P(a)_\alpha C - B \frac{P(a)}{\sin\alpha} m S \right] \right.$$

$$\left. - i\, a_n'' h_n \sum_m M \frac{P(t)}{\sin\theta} \left[ A \frac{P(a)}{\sin\alpha} m^2 C - B\, P(a)_\alpha m S \right] \right\} \hat{\theta}$$

(59b)

$$- \sum_n N \left\{ a_n' \frac{[Krh_n]'}{Kr} \sum_m M \frac{P(t)}{\sin\theta} \left[ B \frac{P(a)}{\sin\alpha} m^2 C + A\, P(a)_\alpha m S \right] \right.$$

$$\left. - i\, a_n'' h_n \sum_m M\, P(t)_\theta \left[ B\, P(a)_\alpha C + A \frac{P(a)}{\sin\alpha} m S \right] \right\} \hat{\varphi}$$

where, for brevity,

$N = 2i^{-n-1} \quad (2n+1)/n(n+1)$

$M = \varepsilon_m (n-m)!/(n+m)!$

$P(t) = P_n^m \,(\cos\theta)$

$P(a) = P_n^m \,(\cos\alpha)$

$C = \cos m\varphi$

$S = \sin m\varphi$ .

Similarly

$$\underline{H}^r = \underline{H}^p + \underline{H}^s$$

$$\underline{H}^{s} = \underline{H}'^{s} + \underline{H}''^{s} = \left(K^2 + \frac{\partial^2}{\partial r^2}\right)(\psi''_s r)\,\hat{r} + \left[-\frac{iK\eta}{\sin\theta}\frac{\partial\psi'_s}{\partial\varphi} + \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}(\psi''_s r)\right]\hat{\theta}$$

(59c)

$$+\left[iK\eta\frac{\partial\psi'_s}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial^2}{\partial r\partial\varphi}(\psi''_s r)\right]\hat{\varphi} \;.$$

Making use of (58) again, we obtain

$$\underline{H}^{s} = -\frac{\eta}{Kr}\sum_{n} Nn(n+1)\, a''_n\, h_n \sum_{m} M\, P(t)\left[B\, P(a)_\alpha C + A\frac{P(a)}{\sin\alpha}\, m\, S\right]\hat{r}$$

$$+\eta\sum_{n} N\left\{ia'_n\, h_n \sum_{m} M\,\frac{P(t)}{\sin\theta}\left[B\,\frac{P(a)}{\sin\alpha}\, m^2\, C + A\, P(a)_\alpha\, m\, S\right]\right.$$

(59d)

$$\left.-a''_n\,\frac{[Krh_n]'}{Kr}\sum_{m} M\, P(t)_\theta\left[B\, P(a)_\alpha C + A\,\frac{P(a)}{\sin\alpha}\, m\, S\right]\right\}\hat{\theta}$$

$$+\eta\sum_{n} N\left\{ia'_n\, h_n \sum_{m} M\, P(t)_\theta\left[A\, P(a)_\alpha\, C - B\,\frac{P(a)}{\sin\alpha}\, m\, S\right]\right.$$

$$\left.-a''_n\,\frac{[Krh_n]'}{Kr}\sum_{m} M\,\frac{P(t)}{\sin\theta}\left[A\,\frac{P(a)}{\sin\alpha}\, m^2\, C - B\, P(a)_\alpha\, m\, S\right]\right\}\hat{\varphi} \;,$$

where $\underline{E}^p$ and $\underline{H}^p$ are given by equation (44). The above constitute the solution to the problem.

For $Kr \gg 1$ and $\gg n$, we can substitute the asymptotic form of the Hankel functions into the above series, i.e., $h_n \rightarrow \frac{i^{-n-1}}{Kr} e^{iKr}$, $[Krh_n]'/Kr \rightarrow ih_n$ . Also if $Ka < 1$ the series are rapidly convergent and may be approximated by their first few terms because of the nature of the dependence of the coefficients $a_n'$, $a_n''$, on Ka, i.e., $a_1' = i2(Ka)^3/3$ , $a_n' = i\left[\frac{2^n\ n!}{(2n)!}\right]^2 \frac{(n+1)(Ka)^{2n+1}}{(2n+1)\ n}$ ;

$$a_1'' = -i(Ka)^3/3 \quad , \quad a_n'' = -\, i\left[\frac{2^n\ n!}{(2n)!}\right]^2 \frac{(Ka)^{2n+1}}{2n+1} \quad .$$

(See reference 1, Appendix IV.)

Hence to the order of $(Ka)^3/Kr$ the scattered quantities may be approximated by

$$(60) \qquad {}^o\psi_s' = -\, i2Ka^3A \cos\theta \sin\alpha\ e^{iKr}/r \quad ,$$

$$ {}^o\psi_s'' = i\eta Ka^3[\, B \sin\theta \cos\varphi \cos\alpha + A \sin\theta \sin\varphi]\, e^{iKr}/r \quad ;$$

and

$$ {}^o\underline{E}^s = -\, e^{iKr}\,(K^2a^3/r)\Big\{[A(2 \sin\theta \sin\alpha + \cos\varphi) - B \sin\varphi \cos\alpha]\,\hat{\theta}$$

$$(61) \qquad -[\, A \cos\theta \sin\varphi + B \cos\theta \cos\varphi \cos\alpha]\hat{\varphi}\Big\} \quad ,$$

$$ {}^o\underline{H}^s = -\eta\, {}^oE^s_\varphi\, \hat{\theta} + \eta\, {}^oE^s_\theta\, \hat{\varphi} \quad ,$$

where the significance of the superscript o is as in (16) and (17).

For the plane of incidence, the xz-plane, we can simplify the problem by considering $\varphi = 0$ and $\theta$ (hitherto considered as always positive) as positive only for $x > 0$ and negative for $x < 0$. We then obtain

$$(43') \quad \underline{E}^{i} = \left\{-A\left[\sin(\theta-\alpha)\,\hat{r} + \cos(\theta-\alpha)\,\hat{\theta}\right] + B\hat{\varphi}\right\} e^{-iKr\cos(\theta-\alpha)} ,$$

$$\underline{H}^{i} = \eta\left\{B\left[\sin(\theta-\alpha)\,\hat{r} + \cos(\theta-\alpha)\,\hat{\theta}\right] + A\hat{\varphi}\right\} e^{-iKr\cos(\theta-\alpha)} ;$$

$$\underline{E}^{p} = \left\{A\left[\sin(\theta+\alpha)\,\hat{r} + \cos(\theta+\alpha)\,\hat{\theta}\right] - B\hat{\varphi}\right\} e^{iKr\cos(\theta+\alpha)} ,$$

$$(44') \quad \underline{H}^{p} = \eta\left\{B\left[\sin(\theta+\alpha)\,\hat{r} + \cos(\theta+\alpha)\,\hat{\theta}\right] + A\hat{\varphi}\right\} e^{iKr\cos(\theta+\alpha)} ;$$

$${}^{o}\underline{E}^{s} = -\,e^{iKr}\,(K^{2}a^{3}/r)\left\{A\,(1+2\sin\alpha\sin\theta)\,\hat{\theta} - B\cos\alpha\cos\theta\,\hat{\varphi}\right\},$$

$$(61') \quad {}^{o}\underline{H}^{s} = \eta\left[-\,{}^{o}E^{s}_{\varphi}\,\theta + {}^{o}E^{s}_{\theta}\,\hat{\varphi}\right].$$

On adding the p and s terms we obtain

$$\underline{E}^{r} = e^{iKr\cos(\theta+\alpha)}\left\{A\sin(\theta+\alpha)\,\hat{r} + A\left[\cos(\theta+\alpha) - Q_{\theta}\,e^{iq}\right]\hat{\theta} - B\left[1 - Q_{\varphi}\,e^{iq}\right]\hat{\varphi}\right\},$$

$$(61'') \quad \underline{H}^{r} = \eta\, e^{iKr\cos(\theta+\alpha)}\left\{B\sin(\theta+\alpha)\,\hat{r} \quad B\left[\cos(\theta+\alpha) - Q_{\varphi}\,e^{iq}\right]\hat{\theta} + A\left[1 - Q_{\theta}\,e^{iq}\right]\hat{\varphi}\right\};$$

where

$$q = 2Kr\sin^{2}\left[(\theta+\alpha)/2\right] ,$$

$$Q_{\theta} = (K^{2}a^{3}/r)\left\{(1+2\sin\alpha\sin\theta) + \mathcal{O}(Ka)^{2} \ldots\right\} ,$$

$$Q = (K^{2}a^{3}/r)\left\{\cos\alpha\cos\theta + \mathcal{O}'(Ka)^{2} \ldots\right\} .$$

(The notation $\mathcal{O}(Ka)^{2}$, $\mathcal{O}'(Ka)^{2}$, merely indicates that the next order terms in the curly brackets are to the order of $(Ka)^{2}$ or that the next order terms in Q are to the order of $(Ka)^{5}$).

These expressions can be readily compared with those obtained for the boss cylindrical case for incidence perpendicular to the axis of the boss (5',6',17',17"). This comparison, as well as further considerations of the implications of the above, will be reserved for the last section of this report.

## B. Distributions of Bosses.

Subject to essentially the same restrictions as for the cylindrical case the asymptotic solution for the single boss will now be extended to obtain approximate solutions to some problems of scattered reflection from certain planar distributions of such bosses. The interactions of the bosses will be neglected as previously and the finite distributions considered will be such that their maximum extent is much less than r sin θ (or less than r if $\alpha = 0$).

The procedure to be followed in this section differs from that employed for the analogous cylindrical problems in that from this point on attention will be restricted to the field vectors and we will no longer be concerned with the Hertz potentials. (The reason for this is that it is the simpler procedure to follow. The potential for a distribution of semicylinders was a simple vector quantity having the same direction at a field point as did the "elementary" potentials it comprised, since the unit vector $\hat{k}$ was common to all the quantities. The potential for a distribution of hemispheres, however, consisting of a superposition of elementary potentials each characterized by a different unit radial vector to a field point--would necessitate the manipulation of quantities possessing components along all three coordinate vectors and would result in no simplification.)

### 1) Patterned Distributions.

Consider $\underline{E}^i$ incident on $2N+1 = \mathcal{M}$ bosses of radius-a whose centers are spaced symmetrically (with spacing-b) along a line in the xy-plane through the origin making an angle $\mu$ with the x-axis (a "ray" of bosses). The scattered components of the field reflected from the ray can be expressed as

$$(62) \qquad \underline{F}^s = \sum_{q=-N}^{N} {}^q\underline{F}^s \, e^{-iqb \cos\mu \sin\alpha} \qquad ;$$

where the exponential phase factor is chosen so that the phase is zero at the origin of the central boss (the reference origin) $q = 0$, and where subject to the single scattering hypothesis the ${}^{q}\underline{F}^{s}$ ($F = E, H$) are obtained from (61) by replacing $r, \theta, \varphi$, by $r_q, \theta_q, \varphi_q$, the coordinates with reference to the q'th boss and similarly for the unit vectors. The procedure we will follow is essentially that of reference 1, III B1, except that for simplicity we will work with the asymptotic solution as in (61) rather than with the general solution.

Now if $\mathcal{M} b \ll r \sin\theta$ we have approximately

$$r_q \approx r - qb \sin\theta \cos(\mu - \varphi) \quad ,$$

(63) $$\cos\theta - \cos\theta_q \approx -\frac{qb}{r} \sin\theta \cos(\mu - \varphi) \cos\theta_q \to \theta_q \approx \theta \quad ,$$

$$\varphi_q \approx \varphi + \frac{qb \sin(\varphi - \mu)}{r_q \sin\theta_q} \to \varphi \quad .$$

Embodying the latter approximations into the ${}^{q}\underline{F}^{s}$ (neglecting $qb \sin\theta \cos(\mu - \varphi) \ll r$ in the denominator and similarly approximating $\hat{\theta}_q$ by $\hat{\theta}$ and $\hat{\varphi}_q$ by $\hat{\varphi}$ ) and evaluating the resulting geometrical progression yields, essentialy as for the cylindrical case, that $\underline{F}^{s} = {}^{0}\underline{F}^{s} \sin\mathcal{M}\Gamma/\sin\Gamma$ and

(64) $$\underline{F}^{r} = \underline{F}^{p} + {}^{0}\underline{F}^{s} \sin\mathcal{M}\Gamma/\sin\Gamma \ , \quad \Gamma = Kb[\sin\alpha \cos\mu + \sin\theta \cos(\mu - \varphi)]/2 \quad .$$

For any distribution of bosses in the xy-plane which can be decomposed into lines at arbitrary angles $\mu$ with the x-axis, each line consisting of a superposition of j rays each characterized by a different number of bosses ($\mathcal{M}$), by a different spacing (b), and different boss radius (a), and all lines and rays having the central element in common, we would proceed by summing over j and $\mu$ and obtain (63) but containing

$$\sum_{\mu} \sum_{j} a^{3}_{j(\mu)} \left[ \left( \sin \eta_{j(\mu)} \Gamma_{j(\mu)} / \sin \Gamma_{j(\mu)} \right) - \eta_{j(\mu)} \right] .$$

(See discussion of expression 21).

When $\underline{E}^{i}$ is incident on a reflection lattice of bosses symmetrically distributed with respect to perpendicular axes x', y', (which have been rotated through an angle $\beta$ from the reference x, y axes) characterized by $\eta'_x$ $b'_{x'}$, $\eta'_y$ $b'_{y'}$, we obtain in analogy to (62)

$$\underline{F}^{s} = \sum_{q_{x'}} \sum_{q_{y'}} {}^{q_{x'} q_{y'}}\underline{F}^{s} \, e^{-iK\rho \cos(\mu' + \beta) \sin \alpha} \quad ; \tag{65}$$

where $\rho = \left[ (q_{z'} \, b_{x'})^{2} + (q_{y'} \, b_{y'})^{2} \right]^{\frac{1}{2}}$ and $\tan \mu' = q_{y'} \, b_{y'} / q_{x'} \, b_{x'}$, and where the approximations to be employed are of the same form as in (63) but with $\mu$ replaced by $\mu' + \beta$ and qb by $\rho$. Hence proceeding as in reference 1,III b 2, yields

$$\underline{F}^{r} = \underline{F}^{p} + {}^{o}\underline{F}^{s} \sin \eta_{x'} \Gamma_{x'} \sin \eta_{y'} \Gamma_{y'} / \sin \Gamma_{x'} \sin \Gamma_{y'} \quad ,$$

(66)

$$\Gamma_{x'} = Kb_{x'} \left[ \cos \beta \sin \alpha + \sin \theta \cos(\varphi - \beta) \right] / 2 \quad ,$$

$$\Gamma_{y'} = Kb_{y'} \left[ - \sin \beta \sin \alpha + \sin \theta \sin(\varphi - \beta) \right] / 2 \quad .$$

For a planar superposition of such distributions with a common central boss we would obtain (64) but containing

$$\sum_{\beta} \sum_{j} a^{3}_{j(\beta)} \left[ \frac{\sin \eta_{x'j(\beta)} \Gamma_{x'j(\beta)} \sin \eta_{y'j(\beta)} \Gamma_{y'j(\beta)}}{\sin \Gamma_{x'j(\beta)} \sin \Gamma_{y'j(\beta)}} - \eta_{j(\beta)} \right]$$

### 2) Uniform Random Distributions.

For $\underline{E}^i$ incident on a random distribution of bosses in the xy-plane characterized by a density function M which is independent of the coordinates so that M dx dy is the number of bosses in area dx dy we proceed in analogy to the above by evaluating

$$\underline{F}^s = M \iint {}^{\rho}\underline{F}^s \, e^{-iK\rho\cos\mu\sin\alpha} \, dx\, dy \qquad (67)$$ ,

where $\rho = (x^2+y^2)^{\frac{1}{2}}$ , $\tan\mu = y/x$ .

### (a) Small Finite Distributions

For finite distributions of maximum extent $\ll r \sin\theta$ we employ essentially the same approximations as in (63) and the resulting simple integrals are readily evaluated as in reference 1, III C 1.

For a ray of length d by unit width we obtain

$$\underline{F}^r = \underline{F}^p + {}^{o}\underline{F}^s \; Md \sin\Gamma/\Gamma \qquad (68)$$ .

($\Gamma$ as in (63) with b replaced by d) .

For a rectangular patch $d_{x'}$ by $d_{y'}$, rotated an angle $\beta$ from the x, y, axes as for the lattice, we obtain

$$\underline{F}^r = \underline{F}^p + {}^{o}\underline{F}^s \; Md_{x'} \, d_{y'} \sin\Gamma_{x'} \sin\Gamma_{y'}/\Gamma_{x'}\Gamma_{y'} \qquad (69)$$

($\Gamma_{x'}$, $\Gamma_{y'}$ , as in (66) with the b's replaced by d's) .

For a circular patch of radius $d_o$ we obtain for the plane of incidence

$$\underline{F}^r = \underline{F}^p + {}^{o}\underline{F}^s \; M\pi d_o^2 \left[ 2\, J_1\,(\Gamma_o)/\Gamma_o \right], \quad \Gamma_o = K\, d_o(\sin\alpha + \sin\theta) \qquad (70)$$ ,

where $\theta$ is $\pm$ for $x \gtrless 0$.

These results may also be extended by summation to a coplanar superposition of such distributions each characterized by a different density function, extent, and boss radius.

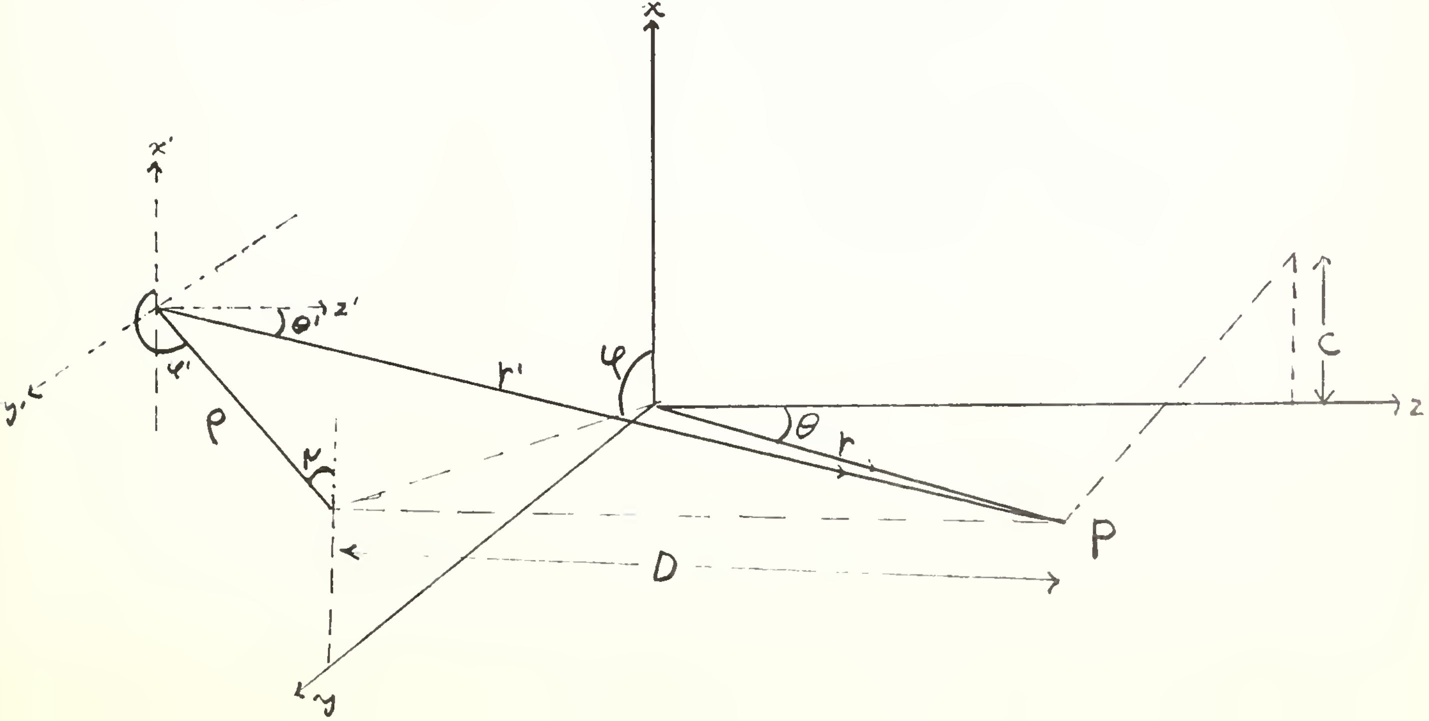
It can be seen that at the specular angle all of the above expressions for the small finite distributions (64, 66, 68, 69, 70)--as well as those for the superposed distributions, provided that all $a_j = a$--can be expressed as

$$(71) \qquad (\underline{F}^r = \underline{F}^p + {}^o\underline{F}^s \eta)_{\theta = \alpha, \varphi = \pi}$$

where $\eta$ is the total number of bosses in the distribution. We may therefore conclude that this is also true, subject to the initial restrictions and to the same order of approximation, for any distribution (of maximum extent $\ll r \sin \theta$) of bosses of equal radii. (See discussion of expression 22).

(b) Infinite Distribution.

For an infinite distribution in the xy plane we evaluate (67) by the method of stationary phase at an arbitrary field point, $z = D = r \cos \theta$, $x = C = r \sin \theta \cos$ , $y = y$; the phase factor in the integral being replaced by $\exp (C + \rho \cos \mu)$ to correspond to the sketch below.



Pursuing the alternate procedure for the cylindrical case, we write

$$(72)\qquad {}^{\rho}\underline{E}^{s} = E_1\,\hat{\theta}' + E_2\,\hat{\varphi}' \quad ,$$

the scalar coefficients $E_1$, $E_2$, (whose values may be obtained from comparison with (61)) being functions of the variables $r'$, $\theta'$, $\varphi'$ the coordinates from a point on the distribution to the field point P. We now transform this quantity and express it in terms of the base unit vectors of the reference frame $\hat{r}, \hat{\theta}, \hat{\varphi}$. Thus

$$\underline{E}^{s} = {}^{\rho}\underline{E}^{s}\cdot\hat{r}\,\hat{r} + {}^{\rho}\underline{E}^{s}\cdot\hat{\theta}\,\hat{\theta} + {}^{\rho}\underline{E}^{s}\cdot\hat{\varphi}\,\hat{\varphi}$$

$$= (E_1\hat{\theta}'\cdot\hat{r} + E_2\hat{\varphi}'\cdot\hat{r})\hat{r} + (E_1\hat{\theta}'\cdot\hat{\theta} + E_2\hat{\varphi}'\cdot\hat{\theta})\,\hat{\theta} + (E_1\hat{\theta}'\cdot\hat{\varphi} + E_2\hat{\varphi}'\cdot\hat{\varphi})\hat{\varphi} \;,$$

where the scalar products may be obtained most readily by expressing both primed and unprimed unit vectors in terms of the reference $\hat{i}$, $\hat{j}$, $\hat{k}$, components by means of the transformation equations given previously with reference to (43). Thus

$$\hat{\theta}'\cdot\hat{r} = (\cos\theta'\cos\varphi'\,\hat{i} + \cos\theta'\sin\varphi'\,\hat{j} - \sin\theta\hat{k})\cdot(\sin\theta\cos\varphi\,\hat{i} + \sin\theta\sin\varphi\,\hat{j} + \cos\theta\,\hat{k})$$

$$= \cos\theta'\sin\theta\cos(\varphi'-\varphi) - \sin\theta'\cos\theta \quad ,$$

$$(74)\quad \hat{\varphi}'\cdot\hat{r} = -\sin\theta\sin(\varphi'-\varphi) \quad ,$$

$$\hat{\theta}'\cdot\hat{\theta} = \cos\theta'\cos\theta\cos(\varphi'-\varphi) + \sin\theta'\sin\theta \;,$$

$$\hat{\varphi}'\cdot\hat{\theta} = -\cos\theta\sin(\varphi'-\varphi) \quad ,$$

$$\hat{\theta}'\cdot\hat{\varphi} = \cos\theta'\sin(\varphi'-\varphi) \quad ,$$

$$\hat{\varphi}'\cdot\hat{\varphi} = \cos(\varphi'-\varphi) \quad .$$

Consequently

$$
\begin{aligned}
\rho\,\underline{E}^s = & \left\{ E_1 [\cos\theta' \sin\theta \cos(\varphi'-\varphi) - \sin\theta' \cos\theta] - E_2 \sin\theta \sin(\varphi'-\varphi) \right\} \hat{r} \\
& + \left\{ E_1 [\cos\theta' \cos\theta \cos(\varphi'-\varphi) \quad \sin\theta' \sin\theta] - E_2 \cos\theta \sin(\varphi'-\varphi) \right\} \hat{\theta} \\
& + \left\{ E_1 \cos\theta' \sin(\varphi'-\varphi) + E_2 \cos(\varphi'-\varphi) \right\} \hat{\varphi} \\
= & - K^2 a^3 (e^{iKr'}/r') \underline{f}(\theta', \varphi') \ .
\end{aligned}
\tag{75}
$$

We can then express (67) as

$$
\underline{E}^s = - K^2 a^3 M \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{iK[r' - (C + \cos\mu)\sin\alpha]} (r')^{-1} \underline{f}(\theta', \varphi') dx\, dy \ ,
\tag{76}
$$

which may be rewritten in terms of θ' and $\mu = \varphi' - \pi$ (as in reference 1, III C2) as

$$
\underline{E}^s = -K^2 a^3 M\, e^{iKC \sin} \int_0^{2\pi} \int_0^{\pi/2} e^{iKD(\sec\theta' - \tan\theta' \sin\alpha \cos\mu)} \times \underline{f}(\theta', \mu+\pi)\, D \tan\theta' \sec\theta'\, d\theta'\, d\mu \ ,
\tag{77}
$$

to yield on evaluation (the phase being stationary at the specular angle, $\theta' = \alpha$, $\mu = 0$, or $\varphi' = \pi$)

$$
\underline{E}^s = - K^2 a^3 M \underline{f}(\theta' = \alpha, \varphi' = \pi)(i2\pi/K) \sec\alpha\, e^{i\underline{K}_p \cdot \underline{r}} \ ,
\tag{78}
$$

which is valid except for the vicinity of $\alpha = \pi/2$.

The same procedure, when applied to obtain $\underline{H}^s$, yields (78) but containing $\underline{f}'$ which differs from the previous $\underline{f}$ only in that $E_1$ is replaced by $-\eta E_2$ and $E_2$ by $\eta E_1$. The quantities $\underline{f}'$ and $\underline{f}$ can then be obtained from (75) by replacing $\theta'$ by $\alpha$ and $\varphi'$ by $\pi$. Hence we obtain the scattered reflected field

$$\underline{E}^r = \underline{E}^p + \underline{E}^s \quad ,$$

$$\underline{E}^s = -i3KV\sec\, e^{i\underline{K}_p\cdot\underline{r}}\Big\{\big[A\cos 2\alpha\,(\cos\alpha\,\sin\theta\cos\varphi + \sin\alpha\cos\theta) - B\cos^2\alpha\,\sin\theta\sin\varphi\big]\,\hat{r}$$

$$+\big[A\cos 2\alpha\,(\cos\alpha\,\cos\theta\cos\varphi - \sin\alpha\,\sin\theta) - B\cos^2\alpha\,\cos\theta\sin\varphi\big]\,\hat{\theta}$$

$$-\big[A\cos 2\alpha\cos\alpha\,\sin\varphi + B\,\cos^2\alpha\,\cos\varphi\big]\,\hat{\varphi}\Big\}$$

$$= -i3KV\sec\alpha\big[\cos 2\alpha\,\underline{E}^p_{\parallel} + \cos^2\alpha\;\;\underline{E}^p_{\perp}\big] \qquad ;$$

(79)

$$\underline{H}^r = \underline{H}^p + \underline{H}^s ,$$

$$\underline{H}^2 = -i\eta 3KV\sec\alpha\, e^{i\underline{K}_p\cdot\underline{r}}\Big\{\big[B\cos^2\alpha\,(\cos\alpha\,\sin\theta\cos\varphi + \sin\alpha\cos\theta) + A\cos 2\alpha\,\sin\theta\sin\varphi\big]\hat{r}$$

$$+\big[B\cos^2\alpha\,(\cos\alpha\,\cos\theta\cos\varphi - \sin\alpha\,\sin\theta) + A\cos 2\alpha\,\cos\theta\sin\varphi\big]\hat{\theta}$$

$$-\big[B\cos^2\alpha\,\cos\alpha\,\sin\varphi - A\cos 2\alpha\,\cos\varphi\big]\hat{\varphi}\Big\}$$

$$= -i3KV\sec\alpha\big[\cos^2\alpha\;\;\underline{H}^p_{\parallel} + \cos 2\alpha\,\underline{H}^p_{\perp}\big] \qquad ;$$

where $V = M\,2\pi\,a^3/3$ is the volumetric departure from the plane; where $\underline{E}^p$ and $\underline{H}^p$ are as in (44); and where the subscripts $\parallel$ and $\perp$ indicate the components $\parallel$ and $\perp$ to the plane of incidence. (Note the relationship of the s and p quantities of these expressions, from which it can be seen that were either A or B equal to zero then the scattered component (s) would be directly proportional to the specular component (p).)

For the plane of incidence, considering $\varphi = 0$ and $\theta$ as $\pm$ for $x \gtrless 0$, the above simplifies to

$$\begin{aligned}
&\underline{E}^r = \underline{E}^p + \underline{E}^s , \\
&\underline{E}^s = -i3KV\sec\alpha\, e^{iKr\cos(\theta+\alpha)} \left\{ A\cos 2\alpha \left[ \sin(\theta+\alpha)\hat{r} + \cos(\theta+\alpha)\hat{\theta} \right] - B\cos^2\alpha\, \hat{\varphi} \right\} ; \\
(79') \quad &\underline{H}^r = \underline{H}^p + \underline{H}^s , \\
&\underline{H}^s = -i\eta 3KV\sec\alpha\, e^{iKr\cos(\theta+\alpha)} \left\{ B\cos^2\alpha \left[ \sin(\theta+\alpha)\hat{r} + \cos(\theta+\alpha)\hat{\theta} \right] + A\cos 2\alpha\, \hat{\varphi} \right\} ,
\end{aligned}$$

where $\underline{E}^p$ and $\underline{H}^p$ are as in (44').

On combining the p and s terms we obtain

$$\begin{aligned}
&\underline{E}^r = \Big\{ A(1 - i3KV\sec\alpha\cos 2\alpha)\left[ \sin(\theta+\alpha)\hat{r} + \cos(\theta+\alpha)\hat{\theta} \right] \\
(79'') \quad &\qquad - B(1 - i3KV\cos\alpha)\hat{\varphi} \Big\}\, e^{iKr\cos(\theta+\alpha)} , \\
&\underline{H}^r = \eta \Big\{ B(1 - i3KV\cos\alpha)\left[ \sin(\theta+\alpha)\hat{r} + \cos(\theta+\alpha)\hat{\theta} \right] \\
&\qquad + A(1 - i3KV\sec\alpha\cos 2\alpha)\hat{\varphi} \Big\}\, e^{iKr\cos(\theta+\alpha)} ;
\end{aligned}$$

which should be compared with the analogous expressions for the cylindrical case, (30"), which they strongly resemble. The discussion of the cylindrical case is also fully applicable to the above.

It should be also noted that $\underline{E}^s$ and $\underline{E}^p$ become formally identical if we substitute $A' = i3KVA \sec\alpha \cos 2\alpha$ and $B' = -i3KVB \cos\alpha$ into the former expression. It can then be seen that the "scattered" Hertz potentials for the infinite distribution are formally identical with those obtained previously for $\underline{E}^p$ provided that we replace A, B, in $\psi'_p, \psi''_p$ as in equation (54) with the above A', B'. No doubt these quantities could have been obtained directly by operating on the ${}^0\psi'_s$, ${}^0\psi''_s$ of expressions (60) but as is apparent this would have been a rather more complicated procedure than the technique employed.

The same qualitative results as above (or as for the cylindrical case for $\beta = \pi/2$) may also be obtained from physical considerations of a very general nature.[12] Consider the reflection of a plane wave, $\underline{E}^i$, from an infinite plane surface covered with an arbitrary distribution of surface irregularities. If the surface were perfectly smooth the reflected wave would be the specularly reflected plane wave, $\underline{E}^p$. The wave reflected from the rough surface would of course not be plane but would be pitted and corrugated in an irregular manner. If, however, the irregularities are assumed to be small compared to the incident wavelength, the reflected wave, having a large specular component, could be considered as essentially plane. We might then express the reflected wave as $\underline{E}^p e^u$ or as $\underline{E}^p (1+u)$; u being a small complex quantity indicating that the reflected wave, although plane, has undergone an amplitude and phase change on reflection. (More correctly we should employ $\underline{E}^p_{\parallel} e^{u_{\parallel}}$, $\underline{E}^p_{\perp} e^{u_{\perp}}$, for the plane of incidence to which the discussion will be restricted. The following should therefore be understood to apply to each of these components separately).

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[12] The above argument will be found in somewhat greater detail in V. Twersky, On the Theory of the Non-Specular Reflection of Sound, pp. 11-14, 64-66, Doctoral Dissertation, New York University, 1949.

This quantity, u, can be but a function of the volume of the irregularities and their density distribution on the plane--or essentially the volumetric departure from the plane, V--the incident wavelength, and the angles of incidence and observation. Hence we write $u = u(\lambda, V, \alpha, \theta)$ which from dimensional considerations yields $u = \gamma KVg(\alpha,\theta)$ where $\gamma$ is a phase factor and g is an even function of $\alpha$ and $\theta$.

(The choice of KV, rather than this quantity raised to some power is justified by there being no a priori physical basis for considering that the scattered component can be anything but a function of the volume of the irregularities. This consideration could of course be obviated by assuming a series in powers of KV and retaining only the first term since we have assumed that $\lambda \gg V$.)

Then by means of essentially the same physical considerations that justify the method of stationary phase (i.e., that the rapid variation in phase for regions not in the vicinity of the specular angle produces destructive interference and virtually complete cancellation between the waves from adjacent elements) we would surmise that practically all the contributions to the scattered reflection were due to the components at the specular angle and write $u = \gamma KVg(\alpha)$ where g is some even function of $\alpha$. Hence we obtain $\underline{E}^r = \underline{E}^p(1 + \gamma KVg(\alpha))$ (for either of the components polarized parallel or perpendicular to the plane of incidence) which is essentially the form of the expressions (79"), (30") derived previously.

C. Distributions of Spheres.

The results of the preceding section can be readily extended to the problems of $\underline{E}^i$ (as in (43)) incident on similar distributions of whole spheres. The expressions for the $\underline{E}^s$ for the small finite distributions (as in (64)-(71))

are formally identical with those that would be obtained for the analogous distributions of spheres if it is understood that ${}^{o}\underline{F}^{s}$ represents the asymptotic solution for the single sphere at the center of the distribution for $Kr \gg 1$, $Ka < 1$.

The solution for the single sphere may be readily obtained by what is essentially a simplification of the previous procedure. Thus from (54) we obtain the forms of $\psi_i$ and consider them to be incident on the sphere. The total potentials are then of the form

$$\psi = \psi_i + \psi_s = \sum (j_n + a_n h_n)\, P_{nm} (-1)^n \quad ,$$

where the $P_{nm}$ are as in (54). Imposing the boundary condition at $r = a$ then yields

$$\psi_s' = \sum a_n' h_n P_{nm}'(-1)^n \quad , \quad \psi_s'' = \sum a_n'' h_n P_{nm}'' (-1)^n \quad ,$$

where the a's are as in (57). The field may be obtained from the above my means of the formulas of (59,a,c).

The rest follows as previously and to the order of $(Ka)^3/Kr$ we obtain

$$\text{(80)} \quad \psi_s' = ie^{iKr}(Ka^3/r)\left[ A(-\cos\theta \sin\alpha + \sin\theta \cos\alpha \cos\varphi) - B \sin\theta \sin\varphi \right] ,$$

$$\psi_s'' = -i\eta e^{iKr}(Ka^3/2r)\left[ B(-\cos\theta \sin\alpha + \sin\theta \cos\alpha \cos\varphi) + A \sin\theta \sin\varphi \right] ;$$

and the field

$$\text{(81)} \quad {}^{o}\underline{E}^{s} = e^{iKr}(K^2a^3/2r)\Big\{\left[-A(2 \sin\theta \sin\alpha + 2 \cos\varphi \cos\theta \cos\alpha + \cos\varphi) + B\sin\varphi(2\cos\theta + \cos\alpha)\right]\hat{\theta}$$

$$+ \left[A \sin\varphi (2 \cos\alpha + \cos\theta) + B(\sin\theta \sin\alpha + \cos\varphi \cos\theta \cos\alpha + 2 \cos\varphi)\right]\hat{\varphi}\Big\} ,$$

$${}^{o}\underline{H}^{s} = \eta(-{}^{o}E^{s}_{\varphi}\hat{\theta} + {}^{o}E^{s}_{\theta}\hat{\varphi}) \quad .$$

Had the incident wave been $\underline{E}^p$ (as in (44)) the above $\psi'_s$ would be modified only to the extent of replacing $\theta$ by $\pi - \theta$ and multiplying ${}^{o}\psi'_s$ by $-1$ and the corresponding changes in the field could be readily determined by means of (59, a, c). It can then be seen that if these two cases are added (i.e., $\underline{E}^i$ and $\underline{E}^p$ simultaneously incident on the sphere, the identical image problem treated for the boss) that the terms in $A \sin\theta \cos\alpha \cos\varphi$ and $-B \sin\theta \sin\varphi$ in the $\psi'_s$ would then cancel leaving only $-2A \cos\theta \sin\alpha$ and the $-B \cos\theta \sin\alpha$ terms in the $\psi''_s$ cancel to leave $A \sin\theta \sin\varphi + B \sin\theta \cos\alpha \cos\varphi$, as was of course expected. (See expressions (60)). To facilitate the extension of the above to the field, and for reference purposes, we will also state the field obtained for $\underline{E}^p$ incident, i.e.,

$$
{}^{o}\underline{E}^s = e^{iKr}(K^2a^3/2r)\Big\{\big[-A(2\sin\theta\sin\alpha - 2\cos\theta\cos\alpha\cos\varphi + \cos\varphi) + B\sin\varphi(\cos\alpha - 2\cos\theta)\big]\hat{\theta} \tag{81'}
$$
$$
+\big[A\sin\varphi(\cos\theta - 2\cos\alpha) + B(\cos\varphi\cos\theta\cos\alpha - 2\cos\varphi - \sin\theta\sin\alpha)\big]\hat{\varphi}\Big\}.
$$

It should be noted from this last expression that if we put $\alpha = 0$, $B = 0$ (to correspond to a wave traveling in the positive z direction and polarized parallel to x) we would obtain

$$
{}^{o}\underline{E}^s = e^{iKr}(K^2a^3/2r)\,A\Big\{\cos\varphi(2\cos\theta - 1)\hat{\theta} + \sin\varphi(\cos\theta - 2)\hat{\varphi}\Big\},
$$

as is also obtained by the standard formulation of the problem of the scattering of a plane electromagnetic wave by a perfectly conducting sphere. (This is included here merely to point out that (81) or (81') could not have been obtained directly from previous formulations of the problem of the sphere--as in reference 10--by a simple transformation of the coordinates such as a rotation around the y-axis but would also require the rotation of the polarization in the xy-plane).

For the infinite distribution and an incident wave $\underline{E}^i$, we obtain by the method of stationary phase applied to (67) utilizing the field of (81), that

$$\underline{E}^{sr} = -i\ KM\pi a^3 \sec\alpha \left[ (1+2\cos 2\alpha)\ \underline{E}^p_{\parallel} + (2+\cos 2\alpha)\ \underline{E}^p_{\perp} \right],$$

(82)

$$\underline{H}^{sr} = -i\ KM\pi a^3 \sec\alpha \left[ (2+\cos 2\alpha)\ \underline{H}^p_{\parallel} + (1+2\cos 2\alpha)\ \underline{H}^p_{\perp} \right],$$

is the reflected wave, and

$$\underline{E}^{st} = i\ KM\pi a^3 \sec\alpha \left[ \underline{E}^i_{\parallel} + \underline{E}^i_{\perp} \right] = i\ KM\pi a^3 \sec\alpha\ \underline{E}^i ,$$

(83)

$$\underline{H}^{st} = i\ KM\pi a^3 \sec\alpha\ \underline{H}^i ,$$

is the transmitted wave (obtained by replacing $\theta'$ by $\pi - \theta'$ in the integral before evaluation, or more directly by obtaining the transmitted wave for $\underline{E}^p$ incident and then replacing the p components by the corresponding i components in the result--since from physical considerations both of the transmitted waves (or both of the reflected waves) or rather their components parallel and perpendicular to the plane of incidence must undergo the same amplitude and phase change).

Had the incident wave been $\underline{E}^p$ we would but interchange the i and p components in the above expressions. As previously, the addition of these two cases serves to check the results derived for the bosses. Thus for z)0 we add (82) (the reflected wave for $\underline{E}^i$ incident) to (83) with $F^i$ replaced by $F^p$ (the transmitted wave for $\underline{E}^p$ incident) and obtain $-1 - 2\cos 2\alpha + 1 = -2\cos 2\alpha$ for the parallel component, and $-2 - \cos 2\alpha + 1 = -2\cos^2\alpha$ for the perpendicular component, as was obtained previously. (See expression (79)).

It should be noted that $4M\pi a^3/3$ is the volume density of spheres or essentially the volume of material per square centimeter of distribution in the xy-plane.

## 5. Discussion and Comparisons.

In this section further consideration will be given to the approximate solutions derived previously. For simplicity and to facilitate comparison the discussion will be restricted to the plane of incidence. **In addition, for the cylindrical case** we will take $\beta = \pi/2$ (incidence perpendicular to the axis of the bosses), while for the spherical case we will adopt the convention that $\varphi = 0$ and $\theta$ is positive or negative depending on $x \gtrless 0$. It is then seen that the corresponding expressions for the two cases resemble each other closely.

Thus from (5'), (6') and (43'),(44') we may write for both sets of image waves,

$$
(84)\qquad
\begin{aligned}
\underline{E}^i &= \{-A[\sin(\theta-\alpha)\hat{r}+\cos(\theta-\alpha)\hat{\theta}] + B\hat{e}\}\, e^{-i\,Kr\cos(\theta-\alpha)} ,\\
\underline{H}^i &= \eta\{B[\sin(\theta-\alpha)\hat{r}+\cos(\theta-\alpha)\hat{\theta}] + A\hat{e}\}\, e^{-i\,Kr\cos(\theta-\alpha)} ,\\
\underline{E}^p &= \{A[\sin(\theta+\alpha)\hat{r}+\cos(\theta+\alpha)\hat{\theta}] - B\hat{e}\}\, e^{i\,Kr\cos(\theta+\alpha)} ,\\
\underline{H}^p &= \eta\{B[\sin(\theta+\alpha)\hat{r}+\cos(\theta+\alpha)\hat{\theta}] + A\hat{e}\}\, e^{i\,Kr\cos(\theta+\alpha)}
\end{aligned}
$$

(where $\hat{e} = \hat{k}$ for the cylindrical case and $\hat{e} = \hat{\varphi}$ for the spherical case). For the scattered components of the reflected field we have for the small finite distributions from (17'), (61'),

$$
(85)\qquad
\begin{aligned}
\underline{E}^c &= -\Delta e^{iKr}(\pi i/2Kr)^{\frac{1}{2}}(Ka)^2[A(1+2\sin\alpha\sin\theta)\hat{\theta} - B2\cos\alpha\cos\theta\ \hat{k}] ,\\
\underline{E}^s &= -\Delta e^{iKr}(K^2a^3/r)[A(1+2\sin\alpha\sin\theta)\,\hat{\theta} - B\cos\alpha\cos\theta\,\hat{\varphi}] ,\\
\underline{H} &= \eta[-F_e\hat{\theta} + F_\theta\hat{e}] ,
\end{aligned}
$$

(where $\Delta$, which yields $\eta$ at the specular angle of reflection, is a "$\Delta$-function" of type and shape that characterizes the particular distribution--see reference 1, p. 29). The "scattered components" for the infinite distributions (henceforth to be indicated by the subscript $\infty$) as in (30'), (79') can be written as

$$
\begin{aligned}
\underline{E}^{c}_{\infty} &= -i\,2KV_c \sec\alpha \left\{ A \cos 2\alpha \left[ \sin(\theta+\alpha)\,\hat{r} + \cos(\theta+\alpha)\,\hat{\theta} \right] - B2 \cos^2\alpha\,\hat{k} \right\} e^{iKr\cos(\theta+\alpha)}, \\
\underline{E}^{s}_{\infty} &= -i\,3KV_s \sec\alpha \left\{ A \cos 2\alpha \left[ \sin(\theta+\alpha)\,\hat{r} + \cos(\theta+\alpha)\,\hat{\theta} \right] - B \cos^2\alpha\,\hat{\varphi} \right\} e^{iKr\cos(\theta+\alpha)}, \\
\underline{H}^{c}_{\infty} &= -i\eta\,2KV_c \sec\alpha \left\{ B\,2\cos^2\alpha \left[ \sin(\theta+\alpha)\,\hat{r} + \cos(\theta+\alpha)\,\hat{\theta} \right] + A \cos 2\alpha\,\hat{k} \right\} e^{iKr\cos(\theta+\alpha)}, \\
\underline{H}^{s}_{\infty} &= -i\eta\,3KV_s \sec\alpha \left\{ B \cos^2\alpha \left[ \sin(\theta+\alpha)\,\hat{r} + \cos(\theta+\alpha)\,\hat{\theta} \right] + A \cos 2\alpha\,\hat{\varphi} \right\} e^{iKr\cos(\theta+\alpha)}.
\end{aligned}
\tag{86}
$$

The expressions for the reflected fields, $\underline{F}^{r} = \underline{F}^{p} + \underline{F}^{\overset{c}{s}}$, consist essentially of two parts. The first (or specularly reflected plane wave component) indicates the effect of the plane, while the second (scattered cylindrical or spherical waves for the single boss and its finite distributions, or plane waves for the infinite distributions) indicates the effect of the bosses. It can be seen that as either a or V $\to$ 0 or $\lambda \to \infty$ (also as r $\to \infty$ for the finite distributions) then $\underline{F}^{r} \to \underline{F}^{p}$ as would be expected from physical considerations.

To the order of approximation of the above expressions it can be seen that the reflected radial component is completely specular while the reflected $\hat{\theta}$ and $\hat{e}$ components possess both specular and non-specular components for the finite distributions and only specular components for the infinite distributions. For all cases, however, the $\hat{\theta}$ and $\hat{e}$ components are reflected with altered amplitude and phase (as is also the $\hat{r}$ component for the infinite distributions) so that the polarization of the reflected wave differs from that of a plane wave undergoing specular reflection at a smooth surface.

The angular dependence of the amplitudes of the scattered fields for the analogous cylindrical and spherical cases resemble each other closely as can be seen from (85) and differ essentially only in the factor 2 possessed by the $\hat{k}$ term. For the infinite distributions, the overall resemblance of the corresponding expressions, although at first startling, might conceivably have been predicted from statistical considerations of both rough and striated surfaces and the similarity of their cross-sections in the plane of incidence--since a rough surface may be approximated by just such a distribution of hemispherical bosses while a striated surface may be approximated by the analogous distribution of parallel semicylinders.

For brevity the reflected fields for the finite distributions may be written as

$$\underline{E}^r = e^{iKr\cos(\theta+\alpha)}\left\{ A\sin(\theta+\alpha)\hat{r} + A\left[\cos(\theta+\alpha) - Q_\theta e^{iq}\right] - B\left[1 - Q_e e^{iq}\right]\hat{e}\right\},$$

(87)

$$\underline{H}^r = \eta\, e^{iKr\cos(\theta+\alpha)}\left\{ B\sin(\theta+\alpha)\hat{r} + B\left[\cos(\theta+\alpha) - Q_e e^{iq}\right]\hat{\theta} + A\left[1 - Q_\theta e^{iq}\right]\hat{e}\right\};$$

where from (17"), (61")

$$q_c = 2Kr\sin^2[(\theta+\alpha)/2] + \pi/4,$$

$$Q^c_\theta = \Delta(\pi/2Kr)^{\frac{1}{2}}(Ka)^2\left\{(1 + 2\sin\alpha\sin\theta) + \mathcal{O}_c(Ka)^2 \ldots\right\},$$

$$Q^c_k = \Delta(\pi/2Kr)^{\frac{1}{2}}(Ka)^2\left\{2\cos\alpha\cos\theta + \mathcal{O}'_c(Ka)^2 \ldots\right\},$$

(88)

$$q_s = 2Kr\sin^2[(\theta+\alpha)/2],$$

$$Q^s_\theta = \Delta(K^2a^3/r)\left\{(1 + 2\sin\alpha\sin\theta) + \mathcal{O}_s(Ka)^2 \ldots\right\},$$

$$Q^s_\varphi = \Delta(K^2a^3/r)\left\{\cos\alpha\cos\theta + \mathcal{O}'_s(Ka)^2 \ldots\right\};$$

while for the infinite distributions

$$\begin{aligned}\underline{E}^r &= e^{iKr\cos(\theta+\alpha)}\left\{ A(1-iK_1)\left[\sin(\theta+\alpha)\hat{r}+\cos(\theta+\alpha)\hat{\theta}\right] - B\left[1-iK_2\right]\hat{e}\right\}, \\ \underline{H}^r &= \eta e^{iKr\cos(\theta+\alpha)}\left\{ B(1-iK_2)\left[\sin(\theta+\alpha)\hat{r}+\cos(\theta+\alpha)\hat{\theta}\right] + A\left[1-iK_1\right]\hat{e}\right\};\end{aligned} \tag{89}$$

where from (30"), (79"),

$$\begin{aligned} K_1^c &= 2\ K\ V_c \sec\alpha\left[\cos 2\alpha + \vartheta_c\ (Ka)^2 \ldots\right], \\ K_2^c &= 4\ K\ V_c \sec\alpha\left[\cos^2\alpha + \vartheta'_c\ (Ka)^2 \ldots\right], \\ K_1^s &= 3\ K\ V_s \sec\alpha\left[\cos 2\alpha + \vartheta_s\ (Ka)^2 \ldots\right], \\ K_2^s &= 3\ K\ V_s \sec\alpha\left[\cos^2\alpha + \vartheta'_s\ (Ka)^2 \ldots\right]. \end{aligned} \tag{90}$$

The above forms will be employed to facilitate the following discussion.

We will now derive expressions for the electromagnetic quantities which are usually measured and for the ratios of the components of these quantities polarized parallel and perpendicular to the plane of incidence. No detailed calculations will be attempted, the aim being but to obtain(for the limiting forms) approximate expressions whose behavior is readily apparent and which are amenable for purposes of calculation.

The reflected intensity (as measured with a non-directional bolometer) is proportional to $J = \mathrm{Re}(\underline{E}^r \cdot \underline{E}^{r*})$ so that for the finite distributions we have

(91) $$J = J_{\parallel} + J_{\perp} = |A|^2\left[1 + Q_\theta^2 - 2\,Q_\theta \cos(\theta+\alpha)\cos q\right] + |B|^2\left[1 + Q_e^2 - 2\,Q_e \cos q\right],$$

while for the infinite distributions

(92) $$J_\infty = J_{\parallel\infty} + J_{\perp\infty} = |A|^2(1 + K_1^2) + |B|^2(1 + K_2^2),$$

where the appropriate quantities may be found in (88) and (90). It can readily be seen that $Q_\theta$ vanishes when $\sin\alpha \sin\theta = -1/2$, so that the component of the reflected radiation polarized parallel to the plane of incidence consists only of the specular contribution from the plane. At the specular angle, $\theta = -\alpha$, this occurs when $\sin^2\alpha = 1/2$ or $\alpha = \pi/4$. For the component polarized perpendicular to the plane of incidence it is seen that $\cos\alpha \cos\theta$ must equal zero (or $\alpha$ or $\theta = \pi/2$) for the scattered contribution to vanish. Similarly for the infinite distributions the scattered component polarized parallel to the plane of incidence vanishes for $\alpha = \pi/4$. The scattered component polarized perpendicular to the plane of incidence cannot legitimately be considered as vanishing at $\alpha = \pi/2$ since this region is excluded from consideration--see reference 1, p. 16, footnote--but since the functions, $K_2$, do not become infinite at $\alpha = \pi/2$ this extension might be accepted for $A = 0$.

It should be noted that although the originally derived expressions were accurate only to the order of $(Ka)^2$ or $(Ka)^3$ for the cylindrical and spherical cases respectively we can justify the retention of both the $Q^2$ and $Q$ terms in the above to only the lowest order for moderately large values of $\Delta$. Thus we write $Q^c = \Delta\left[f_1(Ka)^2 + f_2(Ka)^4 \ldots\right]$ and $(Q^c)^2 = \Delta^2\left[f_1^2(Ka)^4 + 2\,f_1\,f_2\,(Ka)^6 \ldots\right]$ where $f_2$ is substantially less than

$f_1$ (about 1/2 or less)-- and similarly for $Q^s$--so that we may retain only the terms to the order of $\Delta\,(Ka)^2$, $\Delta\,(Ka)^3$, in $Q$ and only the terms $\Delta^2(Ka)^4$, $\Delta^2(Ka)^6$, in $Q^2$ on the assumption that $\Delta^2 f_1^2 (Ka)^4 \gg \Delta f_2 (Ka)^4$, $\Delta^2 f_1^2 (Ka)^6 \gg \Delta f_2 (Ka)^5$. The same considerations may be also applied to the infinite distributions for M large.

The ratios of the reflected intensities polarized parallel and perpendicular to the plane of incidence may be expressed as

$$\rho = \frac{J_{\parallel}}{J_{\perp}} = \frac{|A|^2[1+Q_\theta^2 - 2Q_\theta \cos(\theta+\alpha)\cos q]}{|B|^2[1+Q_e^2 - 2Q_e \cos q]} ,$$

(93)

$$\rho_\infty = \frac{J_{\parallel\infty}}{J_{\perp\infty}} = \frac{|A|^2(1+K_1^2)}{|B|^2(1+K_2^2)} ,$$

where the appropriate quantities are defined in (88), (90).

We will consider only the limiting forms of these expressions. Thus for the finite distributions, if $Q \ll 1$ we neglect $Q^2$ and obtain

$$\rho = (|A|^2/|B|^2)\left\{1-[Q_\theta \cos(\theta+\alpha) - Q_e]\, 2\cos q\right\} ,$$

$$\rho_c = (|A|^2/|B|^2)\Big\{1-\Delta[(1+2\sin\alpha\sin\theta)\cos(\theta+\alpha) - 2\cos\alpha\cos\theta]$$

$$\times (2\pi/Kr)^{\frac{1}{2}}(Ka)^2 \cos\left(2Kr\sin^2[(\alpha+\theta)/2] + \pi/4\right)\Big\} ,$$

(94)

$$\rho_s = (|A|^2/|B|^2)\Big\{1-\Delta[(1+2\sin\alpha\sin\theta)\cos(\theta+\alpha) - \cos\alpha\cos\theta]$$

$$\times (2K^2a^3/r)\cos\left(2Kr\sin^2[(\alpha+\theta)/2]\right)\Big\} ;$$

which are functions of all the parameters of the problem and possess extrema depending on Kr, $\theta$, and $\alpha$. (To plot these quantities as functions of $\theta$ it should be noted that the phase term is very rapidly oscillating, since $Kr \gg 1$, so that it would suffice to consider only its limiting values, $\pm 1$. Similarly the $\Delta$-function, except for its first maximum at $\theta = -\alpha$, may be neglected for an approximate plot). To this order of approximation it can be seen that in addition to the vanishing of the scattered components as determined by the Q's mentioned previously, the scattered contribution parallel to the plane of incidence also vanishes at $\theta + \alpha = \pi/2$.

At the specular angle, $\theta = -\alpha$ and $\Delta \rightarrow \mathcal{M}$ so that the above reduce to

$$\rho_c = (|A|^2/|B|^2)\left[1 + \mathcal{M}(\pi/Kr)^{\frac{1}{2}}(Ka)^2\right], \tag{95}$$

$$\rho_s = (|A|^2/|B|^2)\left[1 + \mathcal{M}\sin^2\alpha \; 2K^2a^3/r\right];$$

Both of these are explicit functions of the number and size of the bosses, the wavelength, and the distance of observation. Only the latter, however, depends on the angle of incidence and has its maximum value at $\alpha = \pi/2$ and its minimum value, which is also its specular value, at $\alpha = 0$. This is a significant difference between the behavior for the cylindrical and spherical cases and its experimental verification should be attempted.

These values also indicate that the maximum values of the $\rho$'s as functions of $\theta$ occur at essentially the specular angle, primarily because of the behavior of the $\Delta$-function.

For the case $Q^2 \gg 1$, as for large $\Delta$ but with $Ka \ll 1$, we obtain, on the assumption that the remaining terms may be neglected, that

$$\rho = (|A|^2/|B|^2)\; Q_\theta^2/Q_e^2 \quad ,$$

(96) $$\rho_c = (|A|^2/|B|^2)\; (1+2\sin\alpha\sin\theta)^2/(2\cos\alpha\cos\theta)^2 \quad ,$$

$$\rho_s = (|A|^2/|B|^2)\; (1+2\sin\alpha\sin\theta)^2/(\cos\alpha\cos\theta)^2 \quad .$$

(It should be noted that the negative square root of the above $\rho_c$ for an unpolarized incident wave ($A = B = 1$), viz.,$-(1+2\sin\alpha\sin\theta/2\cos\alpha\cos\theta)$, was the only quantity considered by Rayleigh[6] with the assumption that the specularly reflected components (due to $\underline{E}^p$) could be screened off--an assumption that might be justified except in the vicinity of the specular angle. Rayleigh found qualitative agreement between the above and experimental results obtained for ruled metallic surfaces or surfaces covered with raised linear protuberances for the cases $\alpha\sim 0$, $\theta\sim\pi/2$ or $\theta\sim 0$, $\alpha\sim\pi/2$ and for the case where $\alpha$ and $\theta$ both made abtuse angles on the same side of the surface normal. He did not, however, find experimentally that the parallel component of the reflected radiation vanishes as $1+2\sin\alpha\sin\theta$ approached zero and surmised that this might be due to the fact that the radii of the protuberances were not small compared to the wavelength, i.e., that the condition $Ka < 1$ had not been satisfied. This could, of course, also be due to the fact that the neglected specular contribution was significant for the angles investigated.) It can be seen that these quantities (96) depend essentially only on the angles of incidence and observation and that the ration of $\rho_c$ to $\rho_s$ is 1/4; or the effect of the hemispherical boss in introducing a relative change of polarization is four times as great as for the semicylindrical boss. It can also be seen that the reflected radiation is polarized perpendicular to the plane of incidence when $1+2\sin\alpha\sin\theta = 0$ and polarized parallel to the plane of incidence when $\cos\alpha\cos\theta = 0$. At the specular angle the above reduce to

(97)
$$\rho_c = (|A|^2/|B|^2)\cos^2 2\alpha/4\cos^4\alpha \quad ,$$
$$\rho_s = (|A|^2/|B|^2)\cos^2 2\alpha/\cos^4\alpha \quad ;$$

so that the parallel component vanishes when cos $2\alpha = 0$ or $\alpha = \pi/4$ and the perpendicular component vanishes when $\cos\alpha = 0$ or $\alpha = \pi/2$. (In view of our comments on Rayleigh's procedure it should be remembered that the specular contributions have been ignored in the statement of (96), (97) so that when the scattered components become small the complete expressions as in (93) or those of (94) should be employed. Consequently the vanishing of a scattered component still leaves the specular component).

For the infinite distributions the second expression of (93) gives some indication of the orders of magnitude to be expected. It can be seen that for $K^2 \gg 1$, as for large M but with $Ka \ll 1$, the expression yields those of (97) which are valid only subject to the above qualifications.

The real part of the radial component of the complex Poynting vector, the average outward energy flow as measured with a directional device, can be expressed as

(98)
$$S = \mathrm{Re}\,(E^r_\theta H^{r*}_e - E^r_e H^{r*}_\theta)/2 = S_\parallel + S_\perp \quad .$$

Hence for the finite distributions

$$S = (\eta/2)\Big\{|A|^2\big(\cos(\theta+\alpha) + Q_\theta^2 - Q_\theta\cos q[1+\cos(\theta+\alpha)]\big)$$
$$+ |B|^2\big(\cos(\theta+\alpha) + Q_e^2 - Q_e\cos q[1+\cos(\theta+\alpha)]\big)\Big\} \quad ,$$

while for the infinite distributions

$$S_{\infty} = (\eta/2)\cos(\theta+\alpha)\left\{|A|^2(1+K_1^2)+|B|^2(1+K_2^2)\right\} , \tag{100}$$

where the quantities are as defined in (88), (90). The behavior of these expressions as regards their dependence on the Q's and K's is the same as that of the J's of (91), (92) and in addition the specular components vanish when $(\theta+\alpha) = \pi/2$.

The ratios of the components polarized parallel and perpendicular to the plane of incidence can be expressed as

$$\sigma = \frac{S_{\parallel}}{S_{\perp}} = \frac{|A|^2\left(\cos(\theta+\alpha)+Q_\theta^2 - Q_\theta \cos q\left[1+\cos(\theta+\alpha)\right]\right)}{|B|^2\left(\cos(\theta+\alpha)+Q_e^2 - Q_e \cos q\left[1+\cos(\theta+\alpha)\right]\right)} \tag{101}$$

for the finite distributions, while for the infinite distributions we obtain $\sigma_{\infty} = \rho_{\infty}$ as in (93).

For the case $Q \ll 1$ we obtain

$$\begin{aligned}
\sigma &= (|A|^2/|B|^2)\left\{1-(Q_\theta - Q_e)\left[1+\sec(\theta+\alpha)\right]\cos q\right\} , \\
\sigma_c &= (|A|^2/|B|^2)\left\{1-\Delta\left[1-2\cos(\theta+\alpha)\right]\left[1+\sec(\theta+\alpha)\right]\right. \\
&\quad \left.\times\ (\pi/2Kr)^{\frac{1}{2}}(Ka)^2\cos\left(2Kr\sin^2[(\theta+\alpha)/2]+\pi/4\right)\right\} , \\
\sigma_s &= (|A|^2/|B|^2)\left\{1-\Delta\left[1-\cos(\theta+\alpha)+\sin\alpha\sin\theta\right]\right. \\
&\quad \left.\times\left[1+\sec(\theta+\alpha)\right](K^2a^3/r)\cos\left(2Kr\sin^2[(\theta+\alpha)/2]\right)\right\} .
\end{aligned} \tag{102}$$

To this order of approximation it can be seen that both these quantities reduce to their specular components when $1+\sec(\theta+\alpha) = 0$ or $\theta+\alpha = \pi$ and that in addition $\sigma_c$ also has its specular value at $1-2\cos(\theta+\alpha) = 0$ or $\theta+\alpha = \pi/3$. Now if $(\theta+\alpha) = \pi/2$ these expressions reduce to

$\sigma = (|A|^2/|B|^2)\ Q_\theta/Q_e$ where $[1+2\sin\alpha\sin(\pi/2-\alpha)]\ /2\cos\alpha\cos(\pi/2-\alpha)$ $= (1+2\sin\alpha\cos\alpha)/2\cos\alpha\sin\alpha = \operatorname{cosec} 2\alpha + 1$. Hence $\sigma_c = (|A|^2/|B|^2)\ (\operatorname{cosec} 2\alpha + 1) = \sigma_s/2$, so that when $\alpha = \pi/4, (\theta = \pi/4)$, $\sigma_c$ reduces to its specular value while $\sigma_s$ is twice its specular value. This is another significant difference between the cylindrical and spherical cases and its experimental verification should be attempted.

It should be noted that the above $\sigma$'s, for positive values of the Q's, exhibit essentially the same behavior as the $\rho$'s in (96) and might therefore account for the qualitative agreement with experiment which Rayleigh found for this range of values--the vanishing of the specular components obviating any additional assumptions as to their significance. These results are also preferable for comparison with Rayleigh's experiments since he employed directional visual devices.

At the specular angle the above quantities (102) become identical with those of (95).

For the case $Q^2 \gg 1$ if we neglect the remaining quantities as previously the expressions for $\sigma$ become identical with those of (96) for $\rho$ subject to the same qualifications.

We will now consider the total reflected intensity and energy flux for the case where the incident wave is unpolarized. (The ratios of the reflected components polarized parallel and perpendicular to the plane of incidence can be readily obtained by setting A = B = 1 in the previous expressions).

For the finite distributions, letting A = B = 1 in (91), (99), we obtain

(103)
$$J = 2 + (Q_\theta^2 + Q_e^2) - 2[\,Q_\theta \cos(\theta+\alpha) + Q_e]\cos q\ ,$$
$$S = (\eta/2)\left\{2\cos(\theta+\alpha) + (Q_\theta^2 + Q_e^2) - (Q_\theta + Q_e)[\,1+\cos(\theta+\alpha)\cos q]\right\}.$$

For the case $Q \ll 1$ we neglect the terms in $Q^2$ and obtain

$$
\begin{aligned}
J_c &= 2\Bigl\{1-\Delta\bigl[(1+2\sin\alpha\sin\theta)\cos(\theta+\alpha)+2\cos\alpha\cos\theta\bigr] \\
&\qquad \times(\pi/2Kr)^{\frac{1}{2}}(Ka)^2\cos\bigl(2Kr\sin^2[(\theta+\alpha)/2]+\pi/4\bigr)\Bigr\}, \\
J_s &= 2\Bigl\{1-\Delta\bigl[(1+2\sin\alpha\sin\theta)\cos(\theta+\alpha)+\cos\alpha\cos\theta \\
&\qquad \times(K^2a^3/r)\cos\bigl(2Kr\sin^2[(\theta+\alpha)/2]\bigr)\Bigr\}, \\
S_c &= \eta\Bigl\{\cos(\theta+\alpha)-\Delta[1+2\cos(\theta-\alpha)][1+\cos(\theta+\alpha)] \\
&\qquad \times(\pi/8Kr)^{\frac{1}{2}}(Ka)^2\cos\bigl(2Kr\sin^2[(\theta+\alpha)/2]+\pi/4\bigr)\Bigr\}, \\
S_s &= \eta\Bigl\{\cos(\theta+\alpha)-\Delta[1+\cos(\theta-\alpha)+\sin\alpha\sin\theta][1+\cos(\theta+\alpha)] \\
&\qquad \times(K^2a^3/2r)\cos\bigl(2Kr\sin^2[(\theta+\alpha)/2]\bigr)\Bigr\}
\end{aligned}
\tag{104}
$$

It should be noted that the latter expressions reduce to $S = -(\eta/2)(Q_\theta+Q_e)\cos q$ when $\theta+\alpha = \pi/2$. Hence the above simplify to $S_c = -\eta\Delta(1+2\sin 2\alpha)\times(\pi/8Kr)^{\frac{1}{2}}(Ka)^2\cos(Kr+\pi/4)$ and $S_s = -\eta\Delta(2+3\sin 2\alpha)(K^2a^3/4r)\cos Kr$, for $\theta = \pi/2-\alpha$.

At the specular angle the expressions of (104) reduce to

$$
\begin{aligned}
S_c &= (\eta/2)\,J_c = \eta\bigl[1-\eta(1+2\cos 2\alpha)(\pi/4Kr)^{\frac{1}{2}}(Ka)^2\bigr], \\
S_s &= (\eta/2)\,J_s = \eta\bigl[1-\eta(1+3\cos 2\alpha)(K^2a^3/2r)\bigr].
\end{aligned}
\tag{105}
$$

These are essentially the extrema of (104) with respect to $\theta$. It should be noted that when $1+2\cos 2\alpha = 0 \rightarrow \alpha = 60^\circ$ and when $1+3\cos 2\alpha = 0 \rightarrow \alpha \approx 55^\circ$, the quantities reduce to their specular values--i.e., for these "critical angles" the reflection at the specular angle is completely specular. Hence for all angles of incidence less than 60°, 55°, for the cylindrical and spherical cases respectively these indicate a minimum in the intensity and energy flux at the specular angle, while for values greater than these critical angles, these quantities have their maximum at the specular angle. The extrema of (105) with respect to $\alpha$ occur at $\sin 2\alpha = 0$ or $\alpha = 0$, $\pi/2$,--the former being the value at the minimum and the latter at the maximum.

Had the incident wave been polarized parallel to the plane of incidence, as obtained by setting A = 1, B = 0, in the initial expressions, then only the $Q_\theta$ term would be present in (104) so that the scattered component of (105) would be proportional to $\cos 2\alpha$ and the critical angle for all the quantities, at which the reflection at the specular angle was completely specular, would occur at $\cos 2\alpha = 0$ or $a = \pi/4$, the expressions having their minimum values below this value and their maximum values above it. The extrema of (105) also occur at $\sin 2\alpha = 0$ or $\alpha = 0, \pi/2$ as above.

Had the incident wave been polarized perpendicular to the plane of incidence, as obtained by setting A = 0, B = 1, in the initial expressions, then only the $Q_e$ term would be present in (104) so that the scattered component of (105) would be proportional to $\cos^2\alpha$ and the critical angle would be $\alpha = \pi/2$ (i.e., the reflected wave at grazing indicence would consist of only the specular component) and the reflected radiation would be a minimum at the specular angle for all other angles of incidence. The extrema of (105) occur when $\cos\alpha \ \sin\alpha = 0$ or $\alpha = 0, \pi/2$, as previously.

For the case $Q^2 \gg 1$, we have on neglecting the Q's that

$$J_c = 2 + \Delta^2[(1+2\sin\alpha\sin\theta)^2 + (2\cos\alpha\cos\theta)^2]\,\pi K^3a^4/2r \quad ,$$

$$J_s = 2 + \Delta^2[(1+2\sin\alpha\sin\theta)^2 + (\cos\alpha\cos\theta)^2]\,K^4a^6/r^2 \quad ,$$

$$(106) \quad S_c = \eta\{\cos(\theta+\alpha) + \Delta^2[(1+2\sin\alpha\sin\theta)^2 + (2\cos\alpha\cos\theta)^2]\,\pi K^3a^4/4r\} ,$$

$$S_s = \eta\{\cos(\theta+\alpha) + \Delta^2[(1+2\sin\alpha\sin\theta)^2 + (\cos\alpha\cos\theta)^2]\,K^4a^6/2r^2\} ;$$

which, at the specular angle, reduce to

$$S_c = (\eta/2)\, J_c = \eta\left\{1 + \eta^2(2\cos^2 2\alpha + 2\cos 2\alpha + 1)\pi K^3 a^4/4r\right\},$$

(107)

$$S_s = (\eta/2)\, J_s = \eta\left\{1 + \eta^2(5\cos^2 2\alpha + 2\cos 2\alpha + 1)\, K^4 a^6/8r^2\right\}.$$

These are essentially the extrema of (106) with respect to θ and indicate a maximum at the specular angle for all angles of incidence--the expressions not reducing to their specular values for any real values of $\alpha$. The extrema of (107) occur when $\sin 2\alpha\,(2\cos 2\alpha + 1) = 0$ or $\alpha = 0, \pi/2, \pi/3$ for the cylindrical case, and when $\sin 2\alpha\,(5\cos 2\alpha + 1) = 0$ or $\alpha = 0, \pi/2$, or $\alpha \approx 51^\circ$ for the spherical case--the expressions possessing maxima at the first two values of $\alpha$ (that at $\alpha = 0$ being the greater) and a minimum, whose value is still greater than the specular value, at the third value of $\alpha$.

Had the incident wave been polarized parallel to the plane of incidence, then (107) would contain $\cos^2 2\alpha$ so that the expressions would reduce to their specular values at $\alpha = \pi/4$, (as above for (105)) but would be the maxima of (106) for all other values of $\alpha$. The extrema of (107) with respect to $\alpha$ would then occur at $2\cos 2\alpha \sin 2\alpha = 0$ or $\alpha = 0, \pi/2, \pi/4$; being maxima for the first two values of $\alpha$ and a minimum (the specular value) for the third.

Had the incident wave been polarized perpendicular to the plane of incidence then (107) would contain $\cos^4\alpha$ so that the expressions would reduce to their specular value at $\alpha = \pi/2$ (as above for (105)) but would be the maxima for all other values of $\alpha$. The extrema of (107) then occur when $4\cos^3\alpha \sin\alpha = 0$ or $\alpha = 0, \pi/2$, being a maximum for the first value of $\alpha$ and a minimum (the specular value) for the second value.

For the infinite distributions we obtain on setting $A = B = 1$ in (92), (100) that

$$J_\infty = 2 + K_1^2 + K_2^2 ,$$

(108)

$$S_\infty = (\eta/2)\cos(\theta+\alpha)(2+K_1^2+K_2^2) .$$

Hence

$$S_c = (\eta/2)\cos(\theta+\alpha)J_c = \eta\cos(\theta+\alpha)\left[1+2\,K^2V_c^2\sec^2\alpha\,(2\cos^2 2\alpha+2\cos 2\alpha+1)\right] ,$$

(109)

$$S_s = (\eta/2)\cos(\theta+\alpha)J_s = \eta\cos(\theta+\alpha)\left[1+(9/8)K^2V_s^2\sec^2\alpha\,(5\cos^2 2\alpha+2\cos 2+1\right]] .$$

At the specular angle, cos $(\theta+\alpha)$ would be replaced by 1, to yield the maximum values for the S's.

It is unfortunate that no quantitative experimental data are at present available with which the above could be compared.* The expressions indicating a minimum in the radiation reflected at the specular angle until some critical angle is reached and a maximum in the reflected radiation above this critical angle are of particular interest and their experimental verification should be attempted. As was mentioned in the introduction, minima at the specular angle had been observed by Skudrzyk[3] when sound underwent non-specular reflection at a rough surface (in the shape of a finite rectangle) whose irregularities were of the same order of magnitude as the wavelength--somewhat substantiating the expressions derived for the acoustical case which indicated essentially the same behavior in this respect.

It should be emphasized that the solutions obtained for the various distributions can only be considered as first order of scattering, or single scattering approximations in the sense that only the primary or plane wave excitation for each boss has been taken into account. Attempts have been made to determine the limits of validity of this approximation and to obtain more accurate solutions. Thus, the problem of the finite grating of parallel cylinders has been formulated to take into account the excitation of each boss caused by the waves scattered (as a result of the primary excitation) by its closest neighbors.[13] It was shown that when $Kb \gg 1$, the largest correction terms are proportional to $(Ka)^4/(Kb)^{1/2}$ so that they vanish more rapidly than the primary excitation terms which are of the order of $(Ka)^2$. (Similarly it can be inferred that for the spherical case, viz., a ray or lattice of spheres, that the largest correction terms would be proportional to $(Ka)^6/Kb$ as compared to $(Ka)^3$ for the primary excitation terms).

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* The writer has just become aware of some work being done by A. Aden and J. Sivak, Cruft Laboratory, Harvard, which might have some bearing on the above.

The above formulation, however, suffers from the fact that the excitation of only the nearest neighbors has been taken into account when as can be seen, from the form of the correction terms and their slow convergence for moderate values of Kb, we should consider the interactions of all the elements of a distribution. To remedy this, formal expressions have been derived for the scattering of a plane wave by any configuration of parallel cylinders which take into account all possible interations of the scattered radiation--i.e., a solution to the scattering problem is obtained which simultaneously satisfies the required boundary conditions at the surface of every cylinder. (An attempt is also being made to treat the analogous problem for a distribution of spheres). These general expressions have been applied to such problems as the grating of parallel cylinders to obtain second order of scattering solutions and will appear in a subsequent report.

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[13] V. Twersky, J. Acoust. Soc. Amer. 22 540 (1950) or see reference 12

# APPENDIX

## Extension to Absorbent Bosses.

Although it has not been found possible to derive solutions to the problems of the boss on the infinite plane when both plane and boss are composed of absorbent material, the analysis of the present paper can be extended to consider the identical problems of non-specular reflection from absorbent bosses on a perfectly conducting plane.

For such surfaces we would replace the previous scattering coefficients $A_n'$, $A_n''$, as in (13)(for the case $\beta = \pi/2$) and $a_n'$, $a_n''$ as in (57) with the scattering coefficients obtained for the problem of the scattering of a plane wave by an absorbent cylinder (for incidence perpendicular to its axis) and by an absorbent sphere respectively. Thus, where the parameters of the absorbent obstacle are expressed by $K_c$, $\mu_c$ and $K_s$, $\mu_s$ respectively, we obtain on imposing the boundary conditions that the tangential components of E and H be continuous across the surface of the obstacle that the new scattering coefficients to be inserted in equation (15') and (59) respectively may be expressed as

$$A_n' = -\frac{\mu_c K\, J_n(K_c a) J_n'(Ka) - \mu K_c J_n'(K_c a) J_n(Ka)}{\mu_c K\, J_n(K_c a) H_n'(Ka) - \mu K_c J_n'(K_c a) H_n(Ka)} ,$$

$$A_n'' = -\frac{\mu_c K\, J_n'(K_c a) J_n(Ka) - \mu K_c J_n(K_c a) J_n'(Ka)}{\mu_c K\, J_n'(K_c a) H_n(Ka) - \mu K_c J_n(K_c a) H_n'(Ka)} ,$$

$$a_n' = -\frac{\mu_s K^2 j_n(Ka)\left[K_s a j_n(K_s a)\right]' - \mu K_s^2 j_n(K_s a)\left[Ka j_n(Ka)\right]'}{\mu_s K^2 h_n(Ka)\left[K_s a j_n(K_s a)\right]' - \mu K_s^2 j_n(K_s a)\left[Ka h_n(Ka)\right]'} ,$$

$$a_n'' = -\frac{\mu j_n(Ka)\left[K_s a j_n(K_s a)\right]' - \mu_s j_n(K_s a)\left[Ka j_n(Ka)\right]'}{\mu h_n(Ka)\left[K_s a j_n(K_s a)\right]' - \mu_s j_n(K_s a)\left[Ka h_n(Ka)\right]'} .$$

It can readily be seen that as $K_c \rightarrow \infty$, as is required for a perfect conductor, that the $A_n$ reduce to the values of (13). Similarly when $K_s \rightarrow \infty$ (remembering that then $j_n(K_s a) \propto (K_s a)^{-1}$) the $a_n$ reduce to the values of (57).

For Ka, $K_{\substack{c\\s}}a < 1$, approximate forms of the above may be obtained by means of the formulas in the appendices of reference 1 and the asymptotic expressions obtained for the single boss can then be readily extended to the distributions discussed in this paper.

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